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Dynamics of K^{th} Order Rational Difference Equation

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Abstract

In this thesis we will investigate the dynamical behavior of the following rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{A + Bx_n + Cx_{n-k}} \qquad n = 0, 1, ... \tag{1}$$

where the parameters α, β, γ and A, B, C and the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are non-negative real numbers, and the denominator is nonzero.

Our concentration here, is on the global stability, the periodic character, the analysis of semi-cycles and the invariant intervals of the positive solution of the above equation.

It is worth to mention that our difference equation is the general case of the rational equation which is studied by Kulenovic and Ladas in their monograph (Dynamics of Second Order Rational Difference Equation with Open Problems and Conjectures, 2002).

الملخص

في هذا البحث سنقوم بدراسة السلوك الديناميكي للمعادلة النسبية المنفصلة التالية:

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{A + Bx_n + Cx_{n-k}} \qquad n = 0,1,...$$

وذلك عندما تكون كل من المتغيرات $\alpha, \beta, \gamma, A, B, C$ والقيم الإبتدائية أعدادًا حقيقة غير سالبة، المتغير k عددًا صحيحًا موجبًا والمقام لا يساوي صفر.

تركيزنا هنا سيكون على ايجاد نقطة الثبات الموجبة للمعادلة السابقة ومعرفة سلوك هذا الحل بناءً على القيم الأولية المدخلة، وسنقوم بدراسة الصفات الدورية وأنصاف الدورات وبعض الخصائص الديناميكية الأخرى للحلول.

تجدر الإشارة هنا إلى أن هذه المعادلة هي الحالة العامة من المعادلة النسبية التي قام كل من Ladas و Kulenvic بدراستها في كتابهم:

[Dynamics of Second Order Rational Difference Equations With Open Problems and Conjectures, 2002].

DECLARATION

I certify that this thesis, submitted for the degree of Master of Science to the Department of Mathematics in Birzeit University, is of my own research expect where otherwise acknowledged, and that this thesis has not been submitted for a higher degree to any other university or institution.

Ayah Zuhair Jasser Asa'd Signature ....

January 26, 2010

Introduction

The dynamical system is the study of the phenomena that evolves in space and / or time by looking at the dynamic behavior or the geometrical and topological properties of the solutions. Whether a particular system comes from biology, physics, chemistry, or even the social sciences, dynamical systems is the subject that provides the mathematical tools for its analysis.

The dynamics of any situation refers to how the situation changes over the course of time. A dynamical system is a physical setting together with rules for how the setting changes or evolves from one moment of time to the next.

In simplest terms, a dynamical system is a system that changes over time. Thus the solar system is a dynamical system, the united state economy is a dynamical system, the weather is a dynamical system, the human heart is a dynamical system.

In mathematics, a dynamical system is a system whose behavior at a given time depends on its behavior at one or more previous time.

There are two types of dynamical system:

1. Differential equations, time is continuous.

2. Difference equations, time is discrete.

In this thesis we will investigate one of the k^{th} order nonlinear difference equations.

This thesis consists of five chapters, chapter one deals with linear and nonlinear difference equations and the solution of these equations, while chapter two deals with the behavior of solutions for difference equations. We will focus on the equilibrium points and their stability, periodic points and the stair step diagrams.

Chapter three is the main one in which we discuss the dynamics of

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{A + Bx_n + Cx_{n-k}}$$

where the parameters $\alpha,\ \beta,\ \gamma$ and A, B, C and the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are non-negative real numbers, and the denominator is nonzero. We will study the local stability, the analysis of semi-cycles and the global stability.

In chapter four we will study the special cases of this equation. Finally, in the last chapter we will present the numerical part of our work.

Contents

1 Solution of Difference Equations

1.1 Introduction to Difference Equation

A difference equation is a sequence of numbers that is generated recursively using a rule to relate each number in the sequence to previous numbers in the sequence. Which means that the term x_{n+1} is related to the terms $x_n, x_{n-1}, \ldots, x_{n-k}$.
This relation expresses itself in the difference equation

$$x_{n+1} = f(x_n) \tag{1.1.1}$$

starting from a point x_0, we may generate the sequence
$x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), \ldots$ for more convenience, we use the notation $f^2(x_0) = f(f(x_0)), \quad f^3(x_0) = f(f(f(x_0))), \ \ldots$
where, $f^2(x_0) = x_2$ is called the second iterate of x_0 under f,
more generally, $f^n(x_0) = x_n$ is called the n^{th} iterate of x_0 under f.

Observe that $x(0) = f^0(x_0) = x_0$
also, $x(n+1) = f^{n+1}(x_0) = f[f^n(x_0)] = f(x_n)$.
Now, let us consider the following difference equation
$x_{n+1} = f(x_n) = (x(n))^2$ for $x_0 = 2, n = 0, 1, 2, \ldots$
The iterated function will produce unbounded orbit $\{2, 4, 16, \ldots\}$.

" The positive orbit $O(x_0)$ of a point x_0 is defined to be the set of points $O(x_0) = \{x_0, f(x_0), f^2(x_0), \ldots\}$ "

1.2 Solution of First Order Linear Difference Equations

In this section we study a special case of Eq. (1.1.1),

$$x_{n+1} = ax_n, \quad a \neq 0 \tag{1.2.1}$$

with initial value x_0 and we will give the details to find the solution and show the behavior of this linear difference equation.

We can calculate the solution of $x(n+1) = ax(n)$ recursively.
Set $x(0) = x_0$.
$x_1 = x(1) = ax(0) = ax_0$
$x_2 = x(2) = ax(1) = a^2x_0$
$x_3 = x(3) = ax(2) = a^3x_0$
$\vdots$
$x_n = x(n) = ax(n-1) = a^nx_0$.

This iterative procedure is an example of a discrete dynamical system.
We make the following results about the limiting behavior of the solution.

1. If $a = 1$, then $\lim_{n\to\infty}(x_n) = x_0$.

2. If $a = -1$, $\lim_{n\to\infty}(x_n) = \begin{cases} x_0 & \text{, if } n \text{ even} \\ -x_0 & \text{, if } n \text{ odd} \end{cases}$

3. If $\mid a \mid > 1$, $\lim_{n\to\infty}(x_n) = \infty$.

4. If $\mid a \mid < 1$, then $\lim_{n\to\infty}(x_n) = 0$.

The previous difference equation (Eq. (1.2.1)) is called a linear homogeneous first order difference equation.
The associated nonhomogeneous equation is given by

$$x_{n+1} = ax_n + b, \qquad a \neq 0$$

where, a and b are real numbers defined for $n \geq 0$.

We get the unique solution of the nonhomogeneous equation by forward iteration with initial condition x_0.
$x_1 = ax_0 + b$.
$x_2 = ax_1 + b = a(ax_0 + b) + b = a^2x_0 + ab + b$.
$x_3 = ax_2 + b = a(ax_1 + b) + b = a^3x_0 + (a^2 + a^1 + 1)b$
$\vdots$
$x_n = ax_{n-1} + b = a(a^{n-1}x_0 + a^{n-2}b + \cdots + b) + b$
$= a^nx_0 + (a^{n-1} + a^{n-2} + \cdots + a + 1)b$.

But the series $a^{n-1} + a^{n-2} + \cdots + a + 1 = \sum_{i=0}^{n-1} a^i$.

And

$$\sum_{i=0}^{n-1} a^i = \begin{cases} n & \text{, if } a = 1 \\ \frac{1-a^n}{1-a} & \text{, if } a \neq 1 \end{cases}$$

Thus the solution of this difference equation is given by

$$x_n = \begin{cases} x_0 + nb & \text{, if } a = 1 \\ a^nx_0 + (\frac{1-a^n}{1-a})b & \text{, if } a \neq 1 \end{cases}$$

Definition 1.2.1. *The order of a dynamical system of difference equation is the difference between the largest and the smallest arguments n appearing in it.*

Example 1.2.1. $x_{n+1} = ax_n + b$, *has order 1.*

$x_{n+4} + ax_n = bx_{n-2}$ *has order 6.*

$x_{n+1} = 2x_n + x_{n-k}$ *has order k+1.*

1.3 Theory of Linear Difference Equations

The normal form of a k^{th} order nonhomogeneous linear difference equation is given by

$$y_{n+k} + p_{k-1}y_{n+k-1} + \cdots + p_1 y_{n+1} + p_0 y_n = g(n) \tag{1.3.1}$$

where $p_i(n)$ and $g(n)$ are real-valued functions defined for $n \geq n_0$
and $p_0(n) \neq 0$.

If $g(n) = 0$ then Eq. (1.3.1) is said to be **a homogeneous equation**.
So, the general form of the k^{th} order homogeneous difference equation is

$$y_{n+k} + p_{k-1}y_{n+k-1} + \cdots + p_1 y_{n+1} + p_0 y_n = 0 \tag{1.3.2}$$

Eq. (1.3.1) may be written in the form

$$y_{n+k} = -p_{k-1}y_{n+k-1} - \cdots + -p_1 y_{n+1} + -p_0 y_n + g(n) \tag{1.3.3}$$

Example 1.3.1. *Consider the second order difference equation*

$$y(n+2) + ny(n+1) - 3y(n) = n \tag{1.3.4}$$

where $y(1) = 0$, $y(2) = -1$,
find the value of y(3), y(4) and y(5).

Solution:
First we rewrite Eq. (1.3.4) in the convenient form.

$y(n+2) = n + -ny(n+1) + 3y(n)$
for $n = 1$ we have $y(3) = 1 + 3y(1) - 1y(2) = 2$
for $n = 2$ $y(4) = 2 + 3y(2) - 2y(3) = -5$
for $n = 3$ $y(5) = 3 + 3y(3) - 3y(4) = 24$

in the same way we can find the other terms of the solutions of our difference equation $y(6), y(7), \ldots$.

A sequence $\{y(n)\}_{n_0}^{\infty}$ or simply $y(n)$ is said to be a solution of Eq. (1.3.1) if it satisfies the equation.

Definition 1.3.1. *[12] The functions $f_1(n),\ f_2(n),\ \ldots,\ f_r(n)$ are said to be linearly independent for $n \geq n_0$, if whenever*

$$a_1 f_1(n) + a_2 f_2(n) + \cdots + a_r f_r(n) = 0$$

for all $n \geq n_0$, then we must have $a_1 = a_2 = \cdots = a_r = 0$.

Example 1.3.2. *Show that the functions 5^n, $n5^n$ and $n^2 5^n$ are linearly independent for $n \geq 1$.*

Solution:
Suppose that for constants c_1 , c_2 and c_3 we have
$c_1 5^n + c_2 n 5^n + c_3 n^2 5^n = 0 \qquad \forall\ n \geq 1,$
$5^n(c_1 + c_2 n + c_3 n^2) = 0$
$\Rightarrow c_1 + c_2 n + c_3 n^2 = 0$ *(dividing by 5^n).*
This is impossible unless $c_3 = 0$, since a second degree equation in n posses at most two solutions.
similarly $c_2 = 0$, whence $c_1 = 0$, which establishes the linear independence of our functions.

Definition 1.3.2. *[12] A set of k linearly independent solutions of Eq. (1.3.2) is called a fundamental set of solutions.*

Theorem 1.3.1. *[12] If $p_0(n) \neq 0$, for all $n \geq n_0$, then the homogeneous difference equation (Eq. (1.3.2)) has a fundamental set of solutions for $n \geq n_0$.*

Definition 1.3.3. *[12] Let $\{x_1(n),\ x_2(n),\ \ldots,\ x_r(n)\}$ be a fundamental set of solutions of Eq. (1.3.2) then the general solution of Eq. (1.3.2) is given by*

$$x(n) = \Sigma_{i=1}^{r} a_i x_i(n)$$

for arbitrary constants a_i.

1.4 Solution of Linear Homogeneous Equations

Consider the K^{th} order difference equation (Eq. (1.3.2))

$$y_{n+k} + p_{k-1}y_{n+k-1} + \cdots + p_1 y_{n+1} + p_0 y_n = 0$$

where the p_i's are constants and $p_0 \neq 0$. Suppose that $y_n = \lambda^n$, where λ is either a complex or real number. Substituting this value into Eq. (1.3.2), we obtain

$$\lambda^k + p_{k-1}\lambda^k + \cdots + p_0 = 0.$$

This equation is called the characteristic equation of Eq. (1.3.2) and its roots $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ are called the characteristic roots. Since $p_0 \neq 0$, so none of the characteristic roots are equal to zero.
There are different cases of λ's, so the general solution of Eq. (1.3.2) has different situations depending on the cases of the characteristic roots.

<u>**Case1:**</u>
Suppose that the characteristic roots $\{\lambda_1,\ \lambda_2,\ \ldots\ \lambda_k\}$ are distinct, and $\{\lambda_1^n,\ \lambda_2^n,\ \ldots\ \lambda_k^n\}$ is a fundamental set of solutions so the general solution is $x(n) = \sum_{i=1}^{k} a_i \lambda_i^n$, a_i are constant numbers.

Example 1.4.1. *Consider the* 2^{nd} *order homogeneous difference equation*
x(n+2)-5x(n+1)+6x(n)=0,   x(0)=0, x(1)=1
find the general solution of this difference equation.

Solution:
The characteristic equation is $\lambda^2 - 5\lambda + 6 = 0$.
Thus, the characteristic roots are $\lambda_1 = 2,\ \lambda_2 = 3$, and these roots give us the general solution $x(n) = c_1(2)^n + c_2(3)^n$.
To find the constants $c_1,\ c_2$ we use the initial values
$x(0) = c_1 + c_2 = 0$
$x(1) = 2c_1 + 3c_2 = 1$
after solving the above system we obtain
$c_1 = -1$ and $c_2 = 1$.
Hence the general solution of the equation is given by
$x(n) = -1(2)^n + 1(3)^n$.

Case2:
Suppose that the characteristic roots $\{\lambda_1,\ \lambda_2,\ \dots\ \lambda_k\}$ all are equal, so the general solution is given by

$$x(n) = \lambda^n(a_0 + a_1 n + \cdots + a_{k-1}n^{k-1})$$

.

Example 1.4.2. *Find the general solution of the following difference equation x(n+2)+8x(n+1)+16x(n)=0.*

Solution:
The characteristic equation of this difference equation is given by
$\lambda^2 + 8\lambda + 16 = 0 \quad \Rightarrow (\lambda + 4)^2 = 0.$
Thus the characteristic roots are $\lambda_1 = \lambda_2 = -4$.
So, the general solution is $x(n) = (c_1 + c_2 n)(-4)^n$.

Case3: Complex characteristic roots.
Assume that the homogeneous difference equation of 2^{nd} order has the complex characteristic roots $\lambda_1 = \alpha + i\beta,\ \lambda_2 = \alpha - i\beta$.
The general solution will be $y(n) = a_1(\alpha + i\beta)^n + a_2(\alpha - i\beta)^n$
In polar coordinate
$\alpha = r\cos\theta,\ \beta = r\sin\theta,\quad r = \sqrt{\alpha^2 + \beta^2},\quad \theta = \tan^{-1}(\frac{\beta}{\alpha})$
So, $x(n) = a_1(r\cos\theta + ir\sin\theta)^n + a_2(r\cos\theta - ir\sin\theta)^n$
By using Moiver's Theorem:
$(r\cos\theta + ir\sin\theta)^n = r^n(\cos(n\theta) + i\sin(n\theta))$
$= r^n((a_1 + a_2)\cos(n\theta) + i(a_1 - a_2)\sin(n\theta))$
$= r^n(c_1\cos(n\theta) + c_2\sin(n\theta))$ where $a_1 + a_2 = c_1$ and $i(a_1 - a_2) = c_2$.
Now let
$\cos\omega = \frac{c_1}{\sqrt{c_1^2+c_2^2}}$ and $\sin\omega = \frac{c_2}{\sqrt{c_1^2+c_2^2}}$
$= r^n\sqrt{c_1^2 + c_2^2}(\cos\omega\cos n\theta + \sin\omega\sin n\theta)$
$=r^n A(\cos(n\theta - \omega))$ where, $A = \sqrt{c_1^2 + c_2^2}$
$\Rightarrow x(n) = r^n A(\cos(n\theta - \omega))$

Example 1.4.3. *Consider the 2^{nd} order homogeneous difference equation x(n+2)+16x(n)=0, write the general solution.*

Solution:
The characteristic equation of the homogeneous equation is $\lambda^2+16 = 0$ which

gives the characteristic roots $\lambda_1 = 0 + 4i$ and $\lambda_2 = 0 - 4i$, thus r=4 and $\theta = \tan^{-1}(\frac{\beta}{\alpha}) = \frac{\pi}{2}$.
So, the general solution is $x(n) = 4^n(c_1 \cos(n\frac{\pi}{2}) + c_2 \sin(n\frac{\pi}{2}))$

1.5 Solution of Nonhomogeneous Linear Equations

In this section we focus our attention on solving the k^{th} order linear nonhomogeneous equations.

$$y_{n+k} + p_{k-1}y_{n+k-1} + \cdots + p_1 y_{n+1} + p_0 y_n = g(n). \tag{1.5.1}$$

Where $p_0 \neq 0$, for all $n \geq n_0$, the sequence $g(n)$ is called the external force, or input of the system.

Example 1.5.1. *Consider the nonhomogeneous difference equation*

$$y(n+2) - y(n+1) - 6y(n) = 5(3)^n \tag{1.5.2}$$

(a) Show that $y_1(n) = n(3)^{n-1}$ and $y_2(n) = (n+1)(3)^{n-1}$ are solutions of the equation.

(b) Show that $y(n) = cn(3)^{n-1}$ is not a solution of the equation, where c is constant.

(c) Show that $y(n) = y_2(n) - y_1(n)$ is not a solution of the equation.

Solution:

(a) To show that $n(3^{n-1})$ is a solution, we substitute $y(n) = n(3^{n-1})$ in the equation $(n+2)3^{n+1} - (n+1)3^n - 6n3^{n-1} = 53^n$
$3^n[3n + 6 - n - 1 - 2n] = 53^n$
$\Rightarrow [3n + 6 - n - 1 - 2n] = 5.$
So $y_1(n) = n(3)^{n-1}$ is a solution of the equation.
In the same way, we see that $y_2(n) = (n+1)(3)^{n-1}$ is a solution of the equation.

(b) To see if $y(n) = cn(3)^{n-1}$ is not a solution of the equation, we substitute $y(n) = cn(3)^{n-1}$ in the equation.
$c(n+2)(3)^{n+1} - c(n+1)3^n - 6cn3^{n-1}$
$= 3^n[3cn + 6c - cn - c - 2cn] = c53^n$.
So, $y(n) = cn(3^{n-1})$ is not a solution.

(c) $y(n) = y_2(n) - y_1(n) = (1+n)3^{n-1} - n3^{n-1} = 3^{n-1}$
Substituting this into the equation yields, so that $y(n) = (3)^{n-1}$ is not a solution.

From the above example we conclude that neither the sum (difference) of two solutions nor a multiple of a solution is a solution. The sum and the difference of two solutions of the nonhomogeneous equation is actually a solution of the associated homogeneous equation.

Theorem 1.5.1. *[12] If $y_1(n)$ and $y_2(n)$ are solutions of Eq.* (1.5.1) *then $y(n) = y_1(n) - y_2(n)$ is a solution of the corresponding homogeneous equation $y_{n+k} + p_{k-1}y_{n+k-1} + \cdots + p_1 y_{n+1} + p_0 y_n = 0$.*

Theorem 1.5.2. *[12] Any solution y(n)of Eq.* (1.5.1) *may be written as*

$$y(n) = y_p(n) + \sum_{i=1}^{k} a_i x_i(n) \ .$$

Where $\sum_{i=1}^{k} a_i x_i(n)$ is the general solution of the homogeneous equation, it is denoted by $y_c(n)$ the complementary solution of the non homogeneous equations, and $y_p(n)$ (the particular solution) is a solution of the nonhomogeneous equations.

The main idea of solving this nonhomogeneous equation (Eq. (1.5.1)) is to find the particular solution $y_p(n)$, in addition to find $y_c(n)$.
There are some techniques to solve the nonhomogeneous equations, and the following example show one of these techniques.

Example 1.5.2. *Solve the difference equation*

$$y(n+2) + 8y(n+1) + 7y(n) = n3^n \tag{1.5.3}$$

Solution:
The characteristic roots of the homogeneous equation are $\lambda_1 = -1,\ \lambda_2 = -7$.
So, $y_c(n) = c_1(-1)^n + c_2(-7)^n$.
To find the particular solution, let $y_p(n) = 3^n(a_0 + a_1 n)$, substituting this relation into Eq. (1.5.3)
We get,
$n3^n = 3^{n+2}(a_0 + a_1(n+2)) + 83^{n+1}(a_0 + a_1(n+1)) + 73^n(a_0 + a_1(n))$
$n3^n = 3^n[9a_0 + 9a_1 n + 18a_1 + 24a_0 + 24a_1 + 24a_1 n + 7a_0 + 7a_1 n]$
hence,
$40a_0 + 42a_1 = 0$ and $40a_1 = 1$
$\Rightarrow a_1 = \frac{1}{40},\ a_0 = \frac{-21}{800}$.
The particular solution is $y_p(n) = 3^n[\frac{-21}{800} + (\frac{1}{40})n]$, and the general solution is $y(n) = y_c(n) + y_p(n)$

So, $y(n) = c_1(-1)^n + c_2(-7)^n + 3^n[\frac{-21}{800} + (\frac{1}{40})n]$.

1.6 Solution of Nonlinear Difference Equations

In general, most nonlinear difference equations cannot be solved explicitly, however some types of them can be solved by transforming nonlinear into linear equations. In this section we study and solve a few types of nonlinear difference equations.

Type I
Equations of Riccati type:

$$x(n+1)x(n) + p(n)x(n+1) + q(n)x(n) = 0 \tag{1.6.1}$$

to solve this equation, let $z(n) = \frac{1}{x(n)}$

we divide Eq. (1.6.1) by $x(n+1)x(n)$, then substituting $z(n) = \frac{1}{x(n)}$
to give us

$$1 + p(n)z(n) + q(n)z(n+1) = 0 \tag{1.6.2}$$

Example 1.6.1. *Solve the difference equation*

$$y(n+1)y(n) - y(n+1) + y(n) = 0$$

$1-\frac{y(n+1)}{y(n+1)y(n)}+\frac{y(n)}{y(n+1)y(n)}=0$
$1-\frac{1}{y(n)}+\frac{1}{y(n+1)}=0$
$1-z(n)+z(n+1)=0, \quad (\ as\ z(n)=\frac{1}{y(n)}\ )$
$z(n+1)=z(n)-1$

and this is first order difference equation.

Type II
Equations of general Riccati Type:

$$x(n+1)=\frac{a(n)x(n)+b(n)}{c(n)x(n)+d(n)} \tag{1.6.3}$$

where $c(n)\neq 0$,
" if $c(n)=0$, then Eq. (1.6.3) will be linear difference equation "
also, $a(n)d(n)-b(n)c(n)\neq 0, \ \forall\ n\geq 0$.

Let $c(n)x(n)+d(n)=\frac{y(n+1)}{y(n)} \Rightarrow x(n)=\frac{y(n+1)}{c(n)y(n)}-\frac{d(n)}{c(n)}$

substitute $x(n)$ in Eq. (1.6.3) to give us

$\frac{y(n+1)}{c(n+1)y(n+1)}-\frac{d(n+1)}{c(n+1)}=\frac{a(n)[\frac{y(n+1)}{c(n)y(n)}-\frac{d(n)}{c(n)}]+b(n)}{\frac{y(n+1)}{y(n)}}.$
This equation simplifies to

$$y(n+2)+p_1(n)y(n+1)+p_2(n)y(n)=0 \tag{1.6.4}$$

where $p_1(n)=\frac{-c(n)d(n+1)+a(n)c(n+1)}{c(n)}$

and $p_2(n)=(a(n)d(n)-b(n)c(n))\frac{c(n+1)}{c(n)}$.

Example 1.6.2. *Solve the difference equation*

$$x(n+1)=\frac{2x(n)+4}{x(n)-1}$$

Solution:
From the above equation we obtain, a= 2, b= 4, c= 1 and d= -1
as $ad-bc=-6\neq 0$ and $c\neq 0$ we will use the transformation

$x(n) - 1 = \frac{y(n+1)}{y(n)}$.
This transformation gives us the following linear difference equation
$y(n+2) - 1y(n+1) - 6y(n) = 0$
the characteristic roots of this equation are $\lambda_1 = 3, \lambda_2 = -2$
hence, $y(n) = c_1(3)^n + c_2(-2)^n$,
but, $x(n) = \frac{y(n+1)}{y(n)} + 1$ by substituting $y(n) = c_1(3)^n + c_2(-2)^n$
in the previous equation we get,

$$x(n) = \frac{c_1(3)^{n+1} + c_2(-2)^{n+1}}{c_1(3)^n + c_2(-2)^n} + 1.$$

Type III
Homogeneous difference equations of the type:

$$f\left(\frac{x(n+1)}{x(n)}, n\right) = 0. \tag{1.6.5}$$

Use the transformation $z(n) = \frac{x(n+1)}{x(n)}$ to transform Eq. (1.6.5) to a linear equation, then we can solve it easily.

Example 1.6.3. *Solve the difference equation*

$$y^2(n+1) - 2y(n+1)y(n) - 3y^2(n) = 0 \tag{1.6.6}$$

Solution:
Divide Eq. (1.6.6) *by* $y^2(n)$
we get,

$$\left(\frac{y(n+1)}{y(n)}\right)^2 - 2\left(\frac{y(n+1)}{y(n)}\right) - 3 = 0 \tag{1.6.7}$$

but $\frac{y(n+1)}{y(n)} = z(n)$, *by substituting it in Eq.* (1.6.7) *we get the following equation*

$$z^2(n) - 2z(n) - 3 = 0$$

$\Rightarrow (z(n) - 3)(z(n) + 1) = 0$

Thus either, z(n)=0 or z(n)=-1
but y(n+1)=z(n)y(n)
So, y(n+1)=3 y(n) or y(n+1)=-y(n)

Type IV

Consider the difference equation of the form

$$(y(n+k))^{r_1}(y(n+k-1))^{r_2}\dots(y(n))^{r_{k+1}} = g(n). \tag{1.6.8}$$

Use the transformation $z(n) = \ln(y(n))$ to convert Eq. (1.6.8) to

$$r_1 z(n+k) + r_2 z(n+k-1) + \cdots + r_{k+1} z(n) = \ln(g(n))$$

Example 1.6.4. *Solve the difference equation*

$$y(n+2) = \frac{y^3(n+1)}{y^2(n)} \tag{1.6.9}$$

Solution:

By taking (ln) for both sides, Eq. (1.6.9) becomes
$\ln y(n+2) = 3\ln y(n+1) - 2\ln y(n)$ let $\ln y(n) = z(n)$.
we obtain,

$$z(n+2) - 3z(n+1) + 2z(n) = 0$$

the characteristic roots of this second order difference equation are $\lambda_1 = 2$, $\lambda_2 = 1$ and the general solution is, $z(n) = c_1(2)^n + c_2(1)^n$
therefore,

$$y(n) = \exp(z(n)) = \exp(c_1(2)^n + c_2).$$

2 Behavior of Solutions for Difference Equations

2.1 The Equilibrium Points

Let us consider the difference equation

$$x(n+1) = f(x_n) \tag{2.1.1}$$

Definition 2.1.1. *[11] A point $\bar{x}$ is said to be an equilibrium point of $x(n+1) = f(x_n)$ if it is a fixed point of f that is $f(\bar{x}) = \bar{x}$.*

Example 2.1.1. *Find the equilibrium points of the following difference equation*

$$x(n+1) = x(n)^2 - 7x(n) + 7$$

Solution:
To find the equilibrium points let $f(\bar{x}) = \bar{x}$.
$\Rightarrow \bar{x} = \bar{x}^2 - 7\bar{x} + 7 \quad \Rightarrow \bar{x}^2 - 8\bar{x} + 7 = 0$
hence there are two equilibrium points $\bar{x} = 7$ and $\bar{x} = 1$

Example 2.1.2. *Determine the fixed points for the equation*

$$f(x) = 5 - \frac{6}{x}$$

Solution:
We can find the fixed points by letting $f(\bar{x}) = \bar{x} \Rightarrow \bar{x} = 5 - \frac{6}{\bar{x}}$
multiplying by $\bar{x}$, we get $\bar{x}^2 - 5\bar{x} + 6 = 0$.
Then we conclude that $\bar{x} = 3$ and $\bar{x} = 2$ are the two fixed points.

Definition 2.1.2. *[12] Let $\mu > 0$, then the difference equation*

$$x(n+1) = \mu x(n)[1 - x(n)] \tag{2.1.2}$$

is called the logistic difference equation and the function

$$F_\mu(x) = \mu x(1 - x)$$

is called logistic map.

Example 2.1.3. *Find the equilibrium points of Eq.* (2.1.2).

Solution:
To find the equilibrium points of the logistic difference equation, we solve the equation $F_\mu(\bar{x}) = \bar{x} \Rightarrow \bar{x} = \mu\bar{x}[1 - \bar{x}]$
hence the two fixed points are 0 and $\frac{\mu-1}{\mu}$.

Graphically
An equilibrium points is the x-coordinate of the point where the graph of f intersects the diagonal line $y = x$.
The two following figures show the equilibrium points of the previous functions.

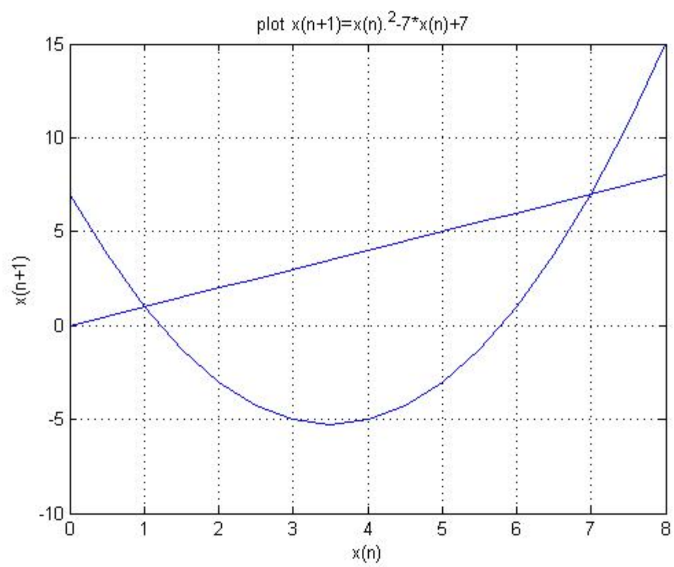

Figure 2.1.0: The equilibrium points of $x(n+1) = x(n)^2 - 7x(n) + 7$

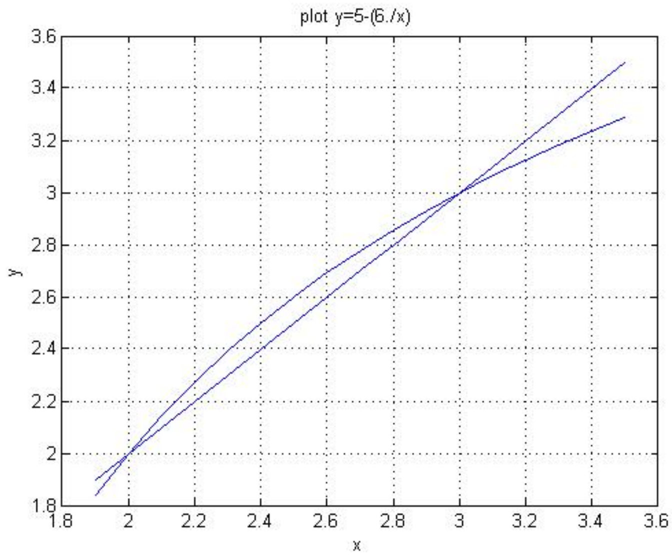

Figure 2.1.1: The fixed points of $f(x) = 5 - \frac{6}{x}$

2.2 Stability Theory

The main objective in the study of dynamical system is to analyze the behavior of its solutions near an equilibrium point, this study constitutes the Stability Theory.

Definition 2.2.1. *[11] Let $\bar{x}$ be an equilibrium point of Eq.* (2.1.1) *and assume that I is some interval of real numbers, where $\bar{x} \in I$ The equilibrium point $\bar{x}$ is called:*

(i) Locally stable " or stable " if for every $\epsilon > 0$, there exists $\delta > 0$ such that for $x_0 \in I$ with $\mid x_0 - \bar{x} \mid < \delta$ we have $\mid x_n - \bar{x} \mid < \in$, for all $n \geq 0$.

(ii) Locally asymptotically stable " or asymptotically stable " if it is locally stable, and if there exists $\gamma > 0$ such that for $x_0 \in I$ with $\mid x_0 - \bar{x} \mid < \gamma$ we have

$$\lim_{n \to \infty} x_n = \bar{x}$$

(iii) A global attractor if for $x_0 \in I$, we have

$$\lim_{n\to\infty} x(n) = \bar{x}$$

(iv) A global asymptotically stable " or globally stable " if it is locally stable and it is a global attractor.

(v) unstable if it is not stable.

(vi) A repeller " or a source " if there exists $r > 0$ such that for $x_0 \in I$ with $\mid x_0 - \bar{x} \mid < r$, there exists $N \geq 1$ such that $\mid x_N - \bar{x} \mid > r$.
Clearly a source is an unstable equilibrium point.

2.3 Criterion For The Asymptotic Stability

In this section, we will state some useful criteria for the asymptotic stability of the equilibrium point.

Theorem 2.3.1. *[12] Let $\bar{x}$ be an equilibrium point of the difference equation*

$$x(n+1) = f(x_n) \tag{2.3.1}$$

where f is continuously differentiable at $\bar{x}$. Then the following statements are true:

(1) If $\mid f'(\bar{x}) \mid < 1$, then $\bar{x}$ is asymptotically stable.

(2) If $\mid f'(\bar{x}) \mid > 1$, then $\bar{x}$ is unstable.

Example 2.3.1. *Consider the difference equation*

$$x(n+1) = x(n)^2 - 7x(n) + 7$$

as we have seen in section (2.1) this equation has two equilibrium points $\bar{x} = 7$ and $\bar{x} = 1$. The equilibrium point $\bar{x} = 7$ is unstable since $f'(7) = 2(7) - 7 = 7 > 1$ so, $\bar{x} = 7$ is unstable as $\mid f'(\bar{x}) \mid > 1$, also, the equilibrium point $\bar{x} = 1$ is unstable, since $\mid f'(1) \mid = \mid 2(1) - 7 \mid = \mid -5 \mid = 5 > 1$.

Example 2.3.2. *Consider the first order difference equation*

$$x(n+1) = x(n)^2 - x(n) + 1.$$

We can easily show that, the only equilibrium point of this equation is $\bar{x} = 1$, *but, when* $\bar{x} = 1$ *then* $| f'(1) |=| 2(1) - 1 |= 1$, *and this case will discuss in the following theorem.*

Theorem 2.3.2. *[12] Suppose that for an equilibrium point* $\bar{x}$ *of Eq.* (2.3.1), $f'(\bar{x}) = 1$ *then the following statements are true:*

(i) If $f''(\bar{x}) \neq 0$, *then* $\bar{x}$ *is unstable.*

(ii) If $f''(\bar{x}) = 0$, *and* $f'''(\bar{x}) > 0$ *then* $\bar{x}$ *is unstable.*

(iii) If $f''(\bar{x}) = 0$, *and* $f'''(\bar{x}) < 0$ *then* $\bar{x}$ *is asymptotically stable.*

Hence, the equilibrium point $\bar{x} = 1$ *from the previous example is unstable as* $f''(1) = 2 \neq 0$.

The preceding Theorem 2.3.2 applied when $f'(\bar{x}) = 1$, and we will use the following theorem in case $f'(\bar{x}) = -1$.
But before stating the theorem, we need to introduce the notation of Schwarzian derivative of a function f, let f be a derivative function then the Schwarzian Sf is given by:

$$Sf(\bar{x}) = \frac{f'''(\bar{x})}{f'(\bar{x})} - \frac{3}{2}\left[\frac{f''(\bar{x})}{f'(\bar{x})}\right]^2$$

and when $f'(\bar{x}) = -1$
then

$$Sf(\bar{x}) = -f'''(\bar{x}) - \frac{3}{2}\left[f''(\bar{x})\right]^2$$

Theorem 2.3.3. *[12] Suppose that for the equilibrium point* $\bar{x}$ *of Eq.* (2.3.1) $f'(\bar{x}) = -1$, *then the following statements hold:*

(i) If $Sf(\bar{x}) > 0$, *then* $\bar{x}$ *is unstable.*

(ii) If $Sf(\bar{x}) < 0$, *then* $\bar{x}$ *is asymptotically stable.*

Example 2.3.3. *Consider the difference equation*

$$x(n+1) = x(n)^2 + 3x(n)$$

determine the stability of the equilibrium points.
Solution:
This equation has two equilibrium points 0 and -2.
By applying Theorem (2.3.1), we conclude that 0 is unstable as
$f'(0) = 2(0) + 3 = 3 > 1$.
But at $\bar{x} = -2, \Rightarrow f'(-2) = -1$, *so we use Theorem (2.3.3) and we obtain*
$Sf(-2) = -f'''(-2) - \frac{3}{2}\left[f''(-2)\right]^2 = 0 - \frac{3}{2}(4) = -6 < 0$.
Thus the equilibrium point (-2) is asymptotically stable as $Sf(-2) < 0$.

Remark:

◇ Theorem (2.3.2) fails if for a fixed point $\bar{x}$, $f'(\bar{x}) = 1$ *and* $f''(\bar{x}) = f'''(\bar{x}) = 0$.

◇ Theorem (2.3.3) fails if $f'(\bar{x}) = -1$ and $Sf(\bar{x}) = 0$.

Example 2.3.4. $f(x) = -x + 2x^2 - 4x^3$ *for fixed point* $\bar{x} = 0$,
$f'(\bar{x}) = -1$ *and* $Sf(\bar{x}) = 0$, *so we cannot use Theorem (2.3.3).*

2.4 Periodic Points and Cycles

One of the most important notation in the study of dynamical systems is to study its periodicity .

Definition 2.4.1. *[12] Let b be in the domain of f, then*

1. *b is called a periodic point of f if for some positive integer k,* $f^k(b) = b$*.*
   A point is k-periodic if it is a fixed point of f^k*, that is b is an equilibrium point of the difference equation* $x(n+1) = f^k(x(n))$*.*
   The periodic orbit of b,O(b)=$\{b, f(b), f^2(b), \ldots, f^{k-1}(b)\}$ *is called a k-cycle.*

2. *b is called eventually k-periodic if for some positive integer m,*
   $f^{m+k}(b) = f^m(b)$*.*

Graphically:
We can find the k-periodic point of a difference equation, by finding the x- coordinate of the point where the graph of f^k meets the diagonal line $y = x$.

Example 2.4.1. *Consider the first order difference equation* $x(n+1) = 2x^2(n)$*, find 2-periodic points.*
Solution:
Let, $f(x(n)) = x(n+1)$*, so* $f(x) = 2x^2$*,*
and as $f^2(x) = f(f(x))$*, then* $f(2x^2) = 2(2x^2)^2 = 8x^4$*.*
By letting $f^2(x) = x$ *we see that the 2-periodic points of our equation are 0 and* $\frac{1}{2}$*.*
We can also plot $f^2(x)$ *to see from the figure the 2-periodic points, by plotting the graph* $f^2(x)$ *and see where it meets the diagonal line* $y = x$*.*

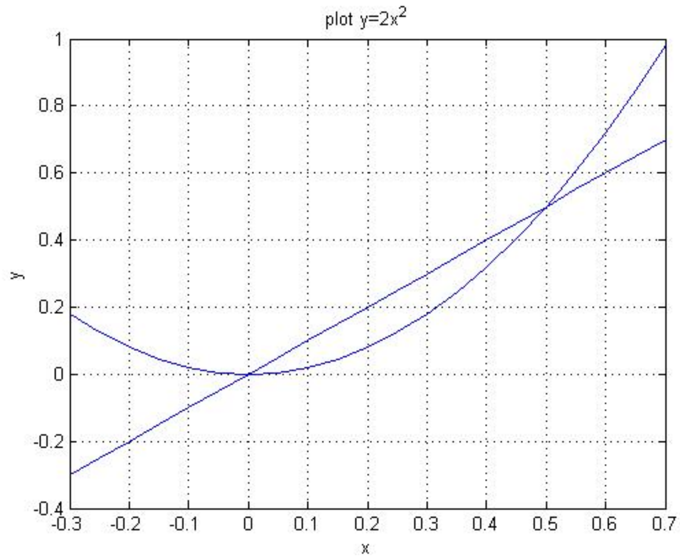

Figure 2.4.1: The 2-periodic points of $x(n+1) = 2x^2(n)$

Definition 2.4.2. *[12] Let b be a k-periodic point of f, then b is*

(1) stable if it is a stable fixed point of f^k.

(2) asymptotically stable if it is an asymptotically stable fixed point of f^k.

(3) unstable if it is an unstable fixed point of f^k.

Theorem 2.4.1. *[12] Let O(b)=$\{b = x(0), x(1), \ldots, x(k-1)\}$ be a k-cycle of a continuously differentiable function f. Then the following statements hold:*

(i) The k-cycle O(b)is asymptotically stable if $\mid f'(x(0))f'(x(1))\ldots f'(x(k-1)) \mid < 1$.

(ii) The k-cycle O(b) is unstable if $\mid f'(x(0))f'(x(1))\ldots f'(x(k-1)) \mid > 1$.

2.5 The Stair Step Diagrams

The stair step diagrams or (cobweb diagram) is a graphical method, for analyzing the stability of equilibrium points for the equation $f(x(n)) = x(n+1)$. We draw a graph of f in the $(x(n), x(n+1))$ plane, and the $y = x$ on the same plane.
We start at an initial point x_0, then we draw a vertical line through x_0 until we intersect the graph of f at (x_0, x_1). Next we draw a horizontal line from $(x_0, x(1))$ to meet the diagonal line $y = x$ at the point $(x(1), x(1))$, then we draw a vertical line from the point $(x(1), x(1))$ to intersect the graph of f at the point $(x(1), x(2))$, and by continuing this process we may find $x(n)$, for all $n > 0$.

Now, we will draw the cobweb diagram around the equilibrium point $\bar{x} = 0$ by taking two initial points $x_0 = .6$ and $x_0 = -.6$, for the function $x(n+1) = 2x^3(n)$.

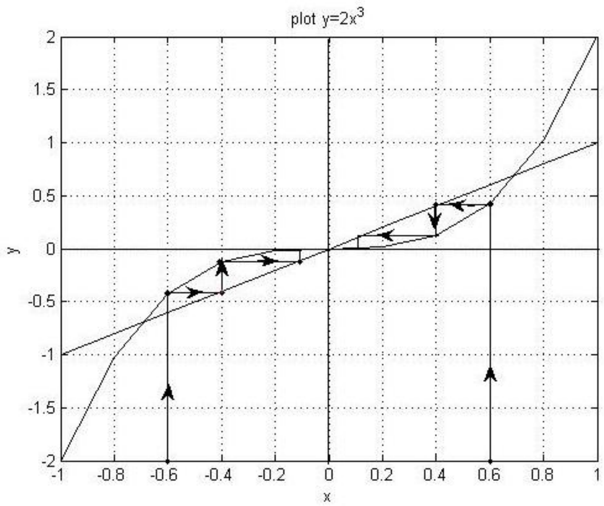

Figure 2.5.1: Stability of $\bar{x} = 0$ of $x(n+1) = 2x^3(n)$

As we can see from this figure, the equilibrium point $\bar{x} = 0$ is asymptotically stable.

2.6 The Limiting Behavior Of The Solutions

To simplify our exposition let us take the second order difference equation

$$y(n+2) + p_1 y(n+1) + p_2 y(n) = 0 \tag{2.6.1}$$

and we study the behavior of its solutions.
Assume that λ_1 and λ_2 are the characteristic roots of the equation. Then we have the following cases.

(a) Case one: Repeated Roots $\lambda_1 = \lambda_2 = \lambda$
The general solution of Eq. (2.6.1) is given by $(a_1 + a_2 n)\lambda^n$.
If $|\lambda| \leq 1$ then the solution $y(n)$ converges to zero.
If $|\lambda| \geq 1$, then the solution $y(n)$ diverges either monotonically if $\lambda \geq 1$ or by oscillating if $\lambda \leq -1$.

(b) Case two: Distinct Roots
Suppose that λ_1 and λ_2 are two real distinct roots, then the general solution of Eq. (2.6.1), as we have seen is given by
$y(n) = a_1\lambda_1^n + a_2\lambda_2^n$.
If $\mid \lambda_1 \mid > \mid \lambda_2 \mid$, we can write $y(n) = \lambda_1^n \left[a_1 + a_2(\frac{\lambda_2}{\lambda_1})^n\right]$
since as $\mid \frac{\lambda_2}{\lambda_1} \mid < 1$ then $\lim_{n\longrightarrow\infty} \left(\frac{\lambda_2}{\lambda_1}\right)^n = 0$,

$\lim_{n\to\infty} y(n) = \lim_{n\to\infty} a_1\lambda_1^n$, there are six different situations depending on the value of λ_1.

1. $\lambda_1 > 1$, The sequence $\{a_1\lambda_1^n\}$ diverges to ∞ " unstable system ".
2. $\lambda_1 = 1$, The sequence $\{a_1\lambda_1^n\}$ is a constant sequence.
3. $0 < \lambda_1 < 1$, The sequence $\{a_1\lambda_1^n\}$ converge to zero " stable system ".
4. $-1 < \lambda_1 < 0$ The sequence $\{a_1\lambda_1^n\}$ is oscillating around zero, and converging to zero " stable system ".
5. $\lambda_1 = -1$ The sequence is oscillating between two values a_1 and $-a_1$.
6. $\lambda_1 < -1$, The sequence $\{a_1\lambda_1^n\}$ is oscillating but increasing in magnitude " unstable system ".

(c) Case three: Two Complex Roots
The last case that we will study here, is when the two roots are complex numbers.
$\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ the general solution of this case as we have seen is $y(n) = Ar^n\cos(n\theta - \omega)$ where, $r = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1}(\frac{\beta}{\alpha})$.
The solution $y(n)$ oscillates since the cosine function oscillates, and this oscillation has three different cases depending on the location of the conjugate characteristic roots:

1. $r = 1$, here λ_1 and $\lambda_2 = \bar{\lambda_1}$ lie on the unit circle in this case $y(n)$ is oscillating but constant in magnitude.
2. $r > 1$, then λ_1, $\lambda_2 = \bar{\lambda_1}$ are outside the unit circle, hence $y(n)$ is oscillating but increasing in magnitude " unstable system ".

3. $r < 1$ then λ_1 and $\lambda_2 = \bar{\lambda_1}$ lie inside the unit disk, the solution $y(n)$ oscillates but converges to zero as $n \longrightarrow \infty$ " stable system ".

We summarize the previous three cases in the following theorem:

Theorem 2.6.1. *[12] The following statements hold.*

(i) All solutions of Eq. (2.6.1) *oscillate about zero if and only if the characteristic equation has no positive real roots.*

(ii) All solutions of Eq. (2.6.1) *converge to zero if and only if* $max\{| \lambda_1 || \lambda_2 |\} < 1$.

Before we state the next theorem, let us consider the second order nonhomogeneous difference equation

$$y(n+2) + p_1 y(n+1) + p_2 y(n) = \mu \tag{2.6.2}$$

where μ is nonzero constant.
The equilibrium point of this equation is $\bar{y} = \frac{\mu}{1+p_1+p_0}$.

So the general solution of Eq. (2.6.2) when $y_p(n) = \bar{y}$ is given by $y(n) = \bar{y} + y_c(n)$.

Now, we can conclude the following theorem.

Theorem 2.6.2. *[12] The following statements hold:*

(i) All solutions of the nonhomogeneous equation (Eq. (2.6.2)) oscillate about the equilibrium solution $\bar{y}$ if and only if none of the characteristic roots of the homogeneous equation (Eq. (2.6.1)) is a positive real number.

(ii) All solutions of Eq. (2.6.2) converge to $\bar{y}$ as $n \longrightarrow \infty$ if and only if max $\{|\lambda_1|, |\lambda_2|\} < 1$ where λ_1 and λ_2 are the characteristic roots of the homogeneous equation (Eq. (2.6.1)).

The previous two theorems give necessary and sufficient conditions under which a second order equation is locally asymptotically stable.
But, the following results provide us with explicit criteria for stability based on the values of the coefficients p_1 and p_2 of Eq. (2.6.1) and Eq. (2.6.2).

Theorem 2.6.3. *[12] The conditions*
$1 + p_1 + p_2 > 0, \quad 1 - p_1 + p_2 > 0, \quad 1 - p_2 > 0$
are necessary and sufficient for the equilibrium point of Eq. (2.6.1) and Eq. (2.6.2) to be asymptotically stable " all solutions converge to $\bar{y}$ ".

Theorem 2.6.4. [1] *The condition*

$$|p_1| < 1 + p_2 < 2$$

is necessary and sufficient for the asymptotically stability of the zero solution of the equation

$$y(n+2) + p_1 y(n+1) + p_2 y(n) = 0$$

Theorem 2.6.5. *[12] Consider the second order difference equation*

$$y(n+2) - p_1 y(n+1) - p_2 y(n) = 0.$$

Then all solutions of the equation converge to zero if $|p_1| + |p_2| < 1$.

[1] See Question 12 in [12], P.97

3 Dynamics of $x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-k}}{A+Bx_n+Cx_{n-k}}$

3.1 Introduction and Preliminaries

Our goal in this chapter is to study the dynamics of the higher order nonlinear difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{A + Bx_n + Cx_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{3.1.1}$$

where the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are non-negative real numbers, $k \in \{1, 2, \ldots\}$, and all the parameters α, β, γ,A, B and C are non-negative real numbers, and the denominator is nonzero.

In 2002, Ladas and Kulenovic in [16] studied the special case of our difference equation, when $k = 1$

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, \qquad n = 0, 1, 2, \ldots$$

where the parameters α, β, γ, A, B and C are non-negative real numbers, and the initial conditions x_{-1}, x_0 are non-negative real numbers, and the denominator is nonzero.

They investigated the local stability, semi-cycles, periodicity, and the invariant intervals.

Li and Sun in [24] studied the dynamical characteristics, such as the global asymptotic stability, the invariant interval, the periodic and oscillatory characters of all positive solutions of the equation

$$x_{n+1} = \frac{px_n + x_{n-k}}{q + x_{n-k}}, \qquad n = 0, 1, 2, \ldots$$

where the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are non-negative real numbers, $k \in \{1, 2, \ldots\}$, and the parameters p and q are non-negative real numbers, and the denominator is nonzero.

Devault et al. in [8] investigated the periodic character and the global stability of the solutions of the difference equation

$$x_{n+1} = \frac{p + x_{n-k}}{qx_n + x_{n-k}}, \qquad n = 0, 1, 2, \ldots$$

where the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are non-negative real numbers, $k \in \{1, 2, \ldots\}$, and all the parameters p and q are non-negative real numbers, and the denominator is nonzero.

Dehghan and Sebdani in [7] investigated the global stability, the boundedness of positive solutions and the character of semi-cycles of the difference equation

$$x_{n+1} = \frac{p + qx_n}{1 + x_{n-k}}, \qquad n = 0, 1, 2, \ldots$$

where the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are non-negative real numbers, $k \in \{1, 2, \ldots\}$, and all the parameters p and q are non-negative real numbers, and the denominator is nonzero.

Also, Dehghan and Douraki in [5] investigated the global stability, invariant intervals and the boundedness of positive solutions of the difference equation

$$x_{n+1} = \frac{p + x_n}{x_n + qx_{n-k}}, \qquad n = 0, 1, 2, \ldots$$

where the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are non-negative real numbers, $k \in \{1, 2, \ldots\}$, and all the parameters p and q are non-negative real numbers, and the denominator is nonzero.

S.Abu Baha in [1] studied the local and global stability, invariant intervals, analysis of semi-cycles and the periodic character of solution of the difference equation

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}, \qquad n = 0, 1, 2, \ldots$$

where the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are non-negative real numbers, $k \in \{1, 2, \ldots\}$, and all the parameters $\beta,\ \gamma,\ B\ \ and\ \ C$ are non-negative real numbers, and the denominator is nonzero.

Farhat and Alaweneh studied independently in [13] and [3] the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}, \qquad n = 0, 1, 2, \ldots$$

where the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are non-negative real numbers, $k \in \{1, 2, \ldots\}$, and all the parameters $\alpha,\ \beta,\ \gamma,\ B\ and\ \ C$ are non-negative real numbers, and the denominator is nonzero.
They studied the periodic character of the positive solution, the invariant intervals, the oscillation and the global stability of all solutions of the above difference equation.

Here, we present the basic definitions and theorems, and some results which will be useful in our investigation of the behavior of solution of Eq. (3.1.1), in this chapter.

Definition 3.1.1. *[19] The equilibrium point $\bar{y}$ of the equation*

$$y_{n+1} = f(y_n, y_{n-1}, \ldots, y_{n-k}), \qquad n = 0, 1, \ldots \tag{3.1.2}$$

is the point that satisfies the condition $\bar{y} = f(\bar{y}, \bar{y}, \ldots, \bar{y})$.

Definition 3.1.2. *[7] Let $\bar{y}$ be an equilibrium point of equation Eq.* (3.1.2), *then the equilibrium point $\bar{y}$ is called:*

1- *Locally stable " or stable " if for every $\epsilon > 0$ there exist $\delta > 0$ such that for all $y_{-k}, \ldots, y_{-1}, y_0 \in I$ with $\sum_{i=-k} \mid y_i - \bar{y} \mid < \delta$ we have $\mid y_n - \bar{y} \mid < \epsilon$ for all $n \geq -k$.*

2- *Locally asymptotically stable" asymptotically stable " if it is locally stable and if there exist $\gamma > 0$ such that for all $y_{-k}, \ldots, y_{-1}, y_0 \in I$ with $\sum_{i=-k} \mid y_i - \bar{y} \mid < \gamma$, we have*
$$\lim_{n \longrightarrow \infty} y_n = \bar{y}.$$

3- *Global attractor if for every $y_{-k}, \ldots, y_{-1}, y_0 \in I$ we have*
$$\lim_{n \longrightarrow \infty} y_n = \bar{y}.$$

4- Globally asymptotically stable if it is locally stable and global attractor.

5- Unstable if it is not stable.

6- A source or a repeller, if there exists $r > 0$ such that for all $y_{-k}, \ldots, y_{-1}, y_0 \in I$ with $\sum_{i=-k} | y_i - \bar{y} |< \gamma$ there exists $N \geq 1$ such that $| y_N - \bar{y} |\geq r$.

The linearized equation associated with Eq. (3.1.2) about the equilibrium point $\bar{y}$ is

$$y_{n+1} = \sum_{i=-k} \frac{\partial f}{\partial u_i}(\bar{y}, \ldots, \bar{y}) y_{n-i} \qquad n = 0, 1, \ldots \tag{3.1.3}$$

and its characteristic equation

$$\lambda^{k+1} = \sum_{i=-k} \frac{\partial f}{\partial u_i}(\bar{y}, \ldots, \bar{y}) \lambda^{k-i}. \tag{3.1.4}$$

3.2 Local Stability

In this section we investigate the locally asymptotic stability of the unique positive equilibrium point of Eq. (3.1.1).

But before investigating the local stability of the positive equilibrium point we utilize the change of variables, let $x_n = \frac{\beta}{B} y_n$ then

$$\frac{\beta}{B} y_{n+1} = \frac{\alpha + \beta \frac{\beta}{B} y_n + \gamma \frac{\beta}{B} y_{n-k}}{A + B\frac{\beta}{B} y_n + C\frac{\beta}{B} y_{n-k}}$$

$\Longrightarrow$

$$\beta y_n = \frac{\alpha B + \beta^2 y_n + \gamma \beta y_{n-k}}{A + \beta y_n + \frac{C\beta}{B} y_{n-k}}$$

then

$$y_{n+1} = \frac{\frac{\alpha B}{\beta} + \beta y_n + \gamma y_{n-k}}{A + \beta y_n + \frac{C\beta}{B} y_{n-k}}$$

hence,

$$y_{n+1} = \frac{\frac{\alpha B}{\beta^2} + y_n + \frac{\gamma}{\beta} y_{n-k}}{\frac{A}{\beta} + y_n + \frac{C}{B} y_{n-k}}.$$

Set $p = \frac{\alpha B}{\beta^2}$, $q = \frac{A}{\beta}$, $L = \frac{\gamma}{\beta}$ and $d = \frac{C}{B}$.

So we get,

$$y_{n+1} = \frac{p + y_n + L y_{n-k}}{q + y_n + d y_{n-k}}. \tag{3.2.1}$$

Let

$$f(x, y) = \frac{p + x + Ly}{q + x + dy}$$

assume that

$$a = \frac{\partial f}{\partial x}(\bar{y}, \bar{y}) \qquad \textit{and} \qquad b = \frac{\partial f}{\partial y}(\bar{y}, \bar{y})$$

$$\frac{\partial f}{\partial x} = \frac{(q + x + dy) - (p + x + Ly)}{(q + x + dy)^2} = \frac{(q - p) + y(d - L)}{(q + x + dy)^2}.$$

So,

$$a = \frac{\partial f}{\partial x}(\bar{y}, \bar{y}) = \frac{(q-p) + \bar{y}(d-L)}{(q+\bar{y}+d\bar{y})^2}$$

and

$$\frac{\partial f}{\partial y} = \frac{L(q+x+dy) - d(p+x+Ly)}{(q+x+dy)^2} = \frac{(Lq-dp) + x(L-d)}{(q+x+dy)^2}.$$

So,

$$b = \frac{\partial f}{\partial x}(\bar{y}, \bar{y}) = \frac{(Lq-dp) + \bar{y}(L-d)}{(q+\bar{y}+d\bar{y})^2}.$$

We notice that the partial derivatives of $f(x,y)$ are evaluated at the equilibrium point $\bar{y}$, so we will find the equilibrium points of Eq. (3.2.1).
Let $f(\bar{y}, \bar{y}) = \bar{y}$ we get,

$$\bar{y} = \frac{p+\bar{y}+L\bar{y}}{q+\bar{y}+d\bar{y}} \Longrightarrow p+\bar{y}+L\bar{y} = q\bar{y} + \bar{y}^2 + d\bar{y}^2$$

$$(1+d)\bar{y}^2 = p - (q-L-1)\bar{y} \tag{3.2.2}$$

we solve Eq. (3.2.2) and find $\bar{y}$

$$\bar{y} = \frac{(L+1-q) \mp \sqrt{(q-L-1)^2 + 4p(1+d)}}{2(d+1)}.$$

The only positive equilibrium point is

$$\bar{y} = \frac{(L+1-q) + \sqrt{(q-L-1)^2 + 4p(1+d)}}{2(d+1)}.$$

For investigation of locally asymptotic stability of the unique positive equilibrium point of Eq. (3.2.1) we need the following theorems:

Theorem 3.2.1. *[17] " Linearized stability "*

1. *If all the roots of Eq.* (3.1.4) *lie in the open unite disk* $|\lambda| < 1$*, then the equilibrium point* $\bar{y}$ *of Eq.* (3.1.2) *is locally stable.*

2. If at least one roots of Eq. (3.1.4) *has absolute value greater than one, then the equilibrium point* $\bar{y}$ *of Eq.* (3.1.2) *is unstable.*

An equilibrium point $\bar{y}$ of Eq. (3.1.2) is a saddle point if there exists a root of Eq. (3.1.4) with absolute value less than one and another root of Eq. (3.1.4)with absolute value greater than one.
An equilibrium point $\bar{y}$ of Eq. (3.1.2) is called a repeller if all roots of Eq. (3.1.4) have absolute value greater than one.

Theorem 3.2.2. *[24] Assume that* $a, b \in \mathbb{R}$ *and* $K \in \{1, 2, \dots\}$ *then*

$$| a | + | b |< 1 \tag{3.2.3}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+1} = ay_n + by_{n-k}, \qquad n = 0, 1, \dots \tag{3.2.4}$$

Suppose in addition that one of the following two cases hold.

(a) K *odd and* $b < 0$.

(b) K *even and* $ab < 0$.

Then Eq. (3.2.3) *is also a necessary condition for the asymptotically stable of Eq.* (3.2.4).

Theorem 3.2.3. *[17] Assume that* $a, b \in \mathbb{R}$. *Then* $| a |< b + 1 < 2$ *is a necessary and sufficient condition for the asymptotically stability of the difference equation*

$$y_{n+1} + ay_n + by_{n-k} = 0, \qquad n = 0, 1, \dots$$

Theorem 3.2.4. *[20] The difference equation*

$$y_{n+1} - by_n + by_{n-k} = 0, \qquad n = 0, 1, \dots$$

is asymptotically stable $iff \ \ 0 <| b |< \frac{1}{2}\cos(\frac{k\pi}{k+2})$.

Note that the linearized equation associated with Eq. (3.2.1) about the equilibrium point $\bar{y}$ is

$$z_{n+1} - az_n - bz_{n-k} = 0.$$

Substitute the values of a and b last in the equation to get,

$$z_{n+1} - \frac{(q-p)+\bar{y}(d-L)}{(q+\bar{y}+d\bar{y})^2} z_n - \frac{(Lq-dp)+(L-d)\bar{y}}{(q+\bar{y}+d\bar{y})^2} z_{n-k} = 0. \qquad (3.2.5)$$

And its characteristic equation is

$$\lambda^{k+1} - \frac{(q-p)+\bar{y}(d-L)}{(q+\bar{y}+d\bar{y})^2}\lambda^k - \frac{(Lq-dp)+(L-d)\bar{y}}{(q+\bar{y}+d\bar{y})^2} = 0.$$

The results presented here and Theorem (3.2.2) give the following theorem

Theorem 3.2.5. *The unique equilibrium point $\bar{y}$ of Eq.* (3.2.1)*is locally asymptotically stable in the following cases:*

1. $d > L$, *there are two cases:*
   (a) $(d-L)\bar{y} < (p-q)$
   (b) $(d-L)\bar{y} > (p-q)$
2. $d < L$, *then we have two cases:*
   (a) $(d-L)\bar{y} < (p-q)$
   (b) $(d-L)\bar{y} > (p-q)$

Proof. We use Theorem (3.2.2),
from the linearized equation we have

$$a = \frac{\bar{y}(d-L)+(q-p)}{(q+\bar{y}+d\bar{y})^2} \quad and \quad b = \frac{\bar{y}(L-d)+(Lq-dp)}{(q+\bar{y}+d\bar{y})^2}$$

1. when $d > L$, there are two cases:

   (a) $(d-L)\bar{y} < (p-q)$ such that $p > q$
   so we have,

$$\mid a \mid = \frac{-\bar{y}(d-L)+(p-q)}{(q+\bar{y}+d\bar{y})^2}, \quad \mid b \mid = \frac{\bar{y}(d-L)+(dp-Lq)}{(q+\bar{y}+d\bar{y})^2} \tag{3.2.6}$$

   we will prove that, $\mid a \mid + \mid b \mid < 1$.

   Substituting the value of a and b,

$$\frac{-\bar{y}(d-L)+(p-q)+\bar{y}(d-L)+(dp-Lq)}{(q+\bar{y}+d\bar{y})^2} < 1.$$

   By multiplying both side with $(q+\bar{y}+d\bar{y})^2$ we get,

$$-\bar{y}(d-L)+p-q+\bar{y}(d-L)+(dp-Lq) < (q+\bar{y}+d\bar{y})^2$$
$$\Longrightarrow p-q+dp-Lq < (q+\bar{y}+d\bar{y})^2.$$

   But

$$(q+\bar{y}+d\bar{y})^2 = (q+(d+1)\bar{y})^2 = q^2+(d+1)^2\bar{y}^2+2q\bar{y}(1+d)$$

   and from Eq. (3.2.2) we get,

$$\bar{y}^2 = \frac{p-(q-L-1)\bar{y}}{(1+d)}.$$

   So,

$$(q+\bar{y}+d\bar{y})^2 = q^2+(d+1)^2\frac{p-(q-L-1)\bar{y}}{(1+d)}+2q\bar{y}(1+d)$$
$$= q^2+q\bar{y}+qd\bar{y}+p+\bar{y}+L\bar{y}+dp+d\bar{y}+dL\bar{y}.$$

Now,

$$p - q + dp - Lq < (q + \bar{y} + d\bar{y})^2$$

$$\Longrightarrow p - q + dp - Lq < q^2 + q\bar{y} + qd\bar{y} + p + \bar{y} + L\bar{y} + dp + d\bar{y} + dL\bar{y}$$

then,

$$0 < q^2 + q\bar{y} + qd\bar{y} + p + \bar{y} + L\bar{y} + dp + d\bar{y} + dL\bar{y}.$$

So, the right hand side is strictly greater than zero.

(b) $(d - L)\bar{y} > (p - q)$ and we have two cases:

i) $p > q$

$$| a |= \frac{\bar{y}(d - L) - (p - q)}{(q + \bar{y} + d\bar{y})^2}, \quad | b |= \frac{\bar{y}(d - L) + (dp - Lq)}{(q + \bar{y} + d\bar{y})^2} \tag{3.2.7}$$

$$\frac{\bar{y}(d - L) - (p - q) + \bar{y}(d - L) + (dp - Lq)}{(q + \bar{y} + d\bar{y})^2} < 1.$$

So,

$$q - p + 2\bar{y}d - 2\bar{y}L + dp - Lq < q^2 + q\bar{y} + qd\bar{y} + p + \bar{y} + L\bar{y} + dp + d\bar{y} + dL\bar{y}$$

cancelling $d\bar{y}$, dp from both sides,
we obtain,

$$\bar{y}d + q < q^2 + q\bar{y} + qd\bar{y} + 2p + \bar{y} + 3L\bar{y} + dL\bar{y}$$

then,

$$0 < (q^2 - q) + q\bar{y} + qd\bar{y} + 2p + \bar{y} + 3L\bar{y} + d\bar{y}(L - 1).$$

This is true only if $q > 1$ and $L \geq 1$.

ii) $p < q$. When $p < q$ *and* $d > L$ then $(dp - Lq) > 0$ or $(dp - Lq) < 0$ if $(dp - Lq) > 0$ then

$$| a |= \frac{\bar{y}(d - L) - (p - q)}{(q + \bar{y} + d\bar{y})^2}, \; | b |= \frac{\bar{y}(d - L) + (dp - Lq)}{(q + \bar{y} + d\bar{y})^2}$$

and this case is the same as (3.2.7), so $| a | + | b |< 1$.

If $(dp - Lq) < 0$ and $\mid (d-L)\bar{y} \mid < \mid dp - Lq \mid$
then we have

$$\mid a \mid = \frac{\bar{y}(d-L)-(p-q)}{(q+\bar{y}+d\bar{y})^2}, \quad \mid b \mid = \frac{-\bar{y}(d-L)-(dp-Lq)}{(q+\bar{y}+d\bar{y})^2} \tag{3.2.8}$$

now we prove that, $\mid a \mid + \mid b \mid < 1$
so,

$$-\bar{y}(d-L) - p + q + \bar{y}(d-L) - dp + Lq < (q+\bar{y}+d\bar{y})^2$$

hence,

$$-p + q - dp + Lq < q^2 + q\bar{y} + qd\bar{y} + p + \bar{y} + L\bar{y} + dp + d\bar{y} + dL\bar{y}$$

$$0 < (q^2 - q) + 2p + 2dp + q\bar{y} + qd\bar{y} + \bar{y} + L\bar{y} + d\bar{y} + dL\bar{y} - Lq$$

then we have,

$$0 < (q^2 - q) + 2p + 2dp + q\bar{y} + \bar{y} + L\bar{y} + d\bar{y} + dL\bar{y} + q(d\bar{y} - L).$$

and this is true only if, $d\bar{y} > L$ and $q > 1$.

2. $d < L$, we have two cases

   (a) $(d-L)\bar{y} < (p-q)$ and there are two subcases

      i) $p < q$

$$\mid a \mid = \frac{-\bar{y}(d-L)+(p-q)}{(q+\bar{y}+d\bar{y})^2}, \quad \mid b \mid = \frac{-\bar{y}(d-L)-(dp-Lq)}{(q+\bar{y}+d\bar{y})^2}. \tag{3.2.9}$$

We will prove that $\mid a \mid + \mid b \mid < 1$
since,

$$-\bar{y}(d-L) + (p-q) + -\bar{y}(d-L) - (dp-Lq) < (q+\bar{y}+d\bar{y})^2$$

hence,

$$-2\bar{y}d + 2L\bar{y} + p - q - dp + Lq < q^2 + q\bar{y} + qd\bar{y} + p + \bar{y} + L\bar{y} + dp + d\bar{y} + dL\bar{y}$$

$$\Longrightarrow 0 < (q^2 + q) + q\bar{y} + qd\bar{y} + \bar{y} - L\bar{y} + 2dp + 3d\bar{y} + dL\bar{y} - Lq$$

thus

$$0 < q(q + 1 + \bar{y} + d\bar{y} - L) + \bar{y} + 2dp + 3d\bar{y} + L(d\bar{y} - \bar{y}).$$

So the right hand side strictly greater than zero.

ii) $p > q$ then $(dp - Lq) > 0$ or $(dp - Lq) < 0$

if $(dp - Lq) > 0$ and $| dp - Lq | > | \bar{y}(d - L) |$

then

$$| a | = \frac{-\bar{y}(d - L) + (p - q)}{(q + \bar{y} + d\bar{y})^2}, \quad | b | = \frac{\bar{y}(d - L) + (dp - Lq)}{(q + \bar{y} + d\bar{y})^2} \tag{3.2.10}$$

and this case is the same as (3.2.6) when $d > L$,
so we have seen that $| a | + | b | < 1$

when $(dp - Lq) > 0$ such that $| (dp - Lq) | < | \bar{y}(d - L) |$ or $(dp - Lq) < 0$,
then we have

$$| a | = \frac{-\bar{y}(d - L) + (p - q)}{(q + \bar{y} + d\bar{y})^2}, \; | b | = \frac{-\bar{y}(d - L) - (dp - Lq)}{(q + \bar{y} + d\bar{y})^2}$$

and this case is the same as (3.2.9), when $d < L$,
so, $| a | + | b | < 1$.

(b) $(d - L)\bar{y} > (p - q)$ such that $p < q$
then

$$| a | = \frac{\bar{y}(d - L) - (p - q)}{(q + \bar{y} + d\bar{y})^2}, \; | b | = \frac{-\bar{y}(d - L) - (dp - Lq)}{(q + \bar{y} + d\bar{y})^2}$$

also, this case is the same as Eq. (3.2.8), so we have seen that

$$| a | + | b | < 1.$$

The proof is complete.

□

Now we will give the following definition which will be the key concept here.

Definition 3.2.1. *[5] An **Invariant Interval** for the difference equation*

$$y_{n+1} = f(y_n, y_{n-1}, y_{n-2}, \ldots, y_{n-k}), \qquad n = 0, 1, \ldots \tag{3.2.11}$$

is an interval I with the property that if k consecutive terms of the solution fall in I, then all subsequent terms of the solution also belong to I. In other words I is an invariant interval for Eq. (3.2.11) *if* $y_{N-k}, \ldots, y_{N-1}, y_N \in I$ *for some* $N \geq 0$ *then* $y_n \in I$, *for every* $n > N$.

Theorem 3.2.6. *Let* $\{y_n\}_{n=-k}^{\infty}$ *be a solution of Eq.* (3.2.1) *then the following are true:*

1. *Suppose that* $L < d$, $p < q$ *and* $dp > Lq$ *and assume that for some* $N \geq 0$ $y_{N-k}, \ldots, y_{N-1}, y_N \in \left[\frac{p+L}{q+d}, 1\right]$ *then* $y_n \in \left[\frac{p+L}{q+d}, 1\right]$, *for all* $n > N$

2. *Suppose that* $L > d$, $p > q$, $dp > Lq$, *and* $\mid Lq - dp \mid > \mid x(L-d) \mid$ *and assume that for some* $N \geq 0$ $y_{N-k}, \ldots, y_{N-1}, y_N \in \left[1, \frac{p+L}{q+d}\right]$ *then* $y_n \in \left[1, \frac{p+L}{q+d}\right]$, *for all* $n > N$

Proof. Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq. (3.2.1)

1. Assume that $L < d$, $p < q$ and $dp > Lq$ then we can easily show that $f(x, y)$ is increasing in x and decreasing in y, by using partial derivative [2]

$$\frac{\partial f(x,y)}{\partial x} = \frac{(q+dy)-(p+Ly)}{(q+x+dy)^2}$$

when $L < d$ and $p < q$ then $\frac{\partial f(x,y)}{\partial x} > 0$, so $f(x, y)$ is increasing in x.

Also,

$$\frac{\partial f(x,y)}{\partial y} = \frac{L(q+x)-d(p+x)}{(q+x+dy)^2}$$

when $L < d$ and $dp > Lq$ then $\frac{\partial f(x,y)}{\partial y} < 0$, and so $f(x, y)$ is decreasing in y.

Now, for some $N > 0$, and $\frac{p+L}{q+d} \leq y_{N-k}, \ldots, y_{N-1}, y_N \leq 1$, we can say that the following step is true as " $p < q$ and $L < d$ "

$$y_{N+1} = \frac{p+y_N+Ly_{N-k}}{q+y_N+dy_{N-k}} \leq \frac{q+y_N+Ly_{N-k}}{q+y_N+dy_{N-k}} \leq \frac{q+y_N+dy_{N-k}}{q+y_N+dy_{N-k}} = 1.$$

So,

$$y_{N+1} \leq 1.$$

And to show that $y_{N+1} \geq \frac{p+L}{q+d}$ we will substitute $y_{N-k} = 1$ and $y_N = \frac{p+L}{q+d}$ in the following function,

$$y_{N+1} = \frac{p+y_N+Ly_{N-k}}{q+y_N+dy_{N-k}}$$

and since y_{N+1} is increasing in y_N and decreasing in y_{N-k}, we get the following,

$$y_{N+1} = \frac{p+y_N+Ly_{N-k}}{q+y_N+dy_{N-k}} \geq \frac{p+\frac{p+L}{q+d}+L(1)}{q+\frac{p+L}{q+d}+d(1)} = \frac{(p+L)\left[1+\frac{1}{q+d}\right]}{(q+d)\left[1+\frac{1}{q+d}\frac{p+L}{q+d}\right]}$$

[2]See Theorem 4.2.2 in [18], P.144

but $\frac{p+L}{q+d} < 1.$

So,

$$\frac{(p+L)\left[1+\frac{1}{q+d}\right]}{(q+d)\left[1+\frac{1}{q+d}\frac{p+L}{q+d}\right]} > \frac{p+L}{q+d}\left[\frac{1+\frac{1}{q+d}}{1+\frac{1}{q+d}}\right] = \frac{p+L}{q+d}$$

then

$$y_{N+1} \geq \frac{p+L}{q+d}$$

By Mathematical Induction we can prove that $y_n \in \left[\frac{p+L}{q+d}, 1\right]$, for all $n > N$. We proved that $y_{N+1} \in \left[\frac{p+L}{q+d}, 1\right]$, so we just will show that if $y_{N+m-1} \in \left[\frac{p+L}{q+d}, 1\right]$ then $y_{N+m} \in \left[\frac{p+L}{q+d}, 1\right]$.

$$y_{N+m} = \frac{p+y_{N+m-1}+Ly_{N+m-(k+1)}}{q+y_{N+m-1}+dy_{N+m-(k+1)}} \leq \frac{q+y_{N+m-1}+Ly_{N+m-(k+1)}}{q+y_{N+m-1}+dy_{N+m-(k+1)}}$$

also,

$$\frac{q+y_{N+m-1}+Ly_{N+m-(k+1)}}{q+y_{N+m-1}+dy_{N+m-(k+1)}} \leq \frac{q+y_{N+m-1}+dy_{N+m-(k+1)}}{q+y_{N+m-1}+dy_{N+m-(k+1)}} = 1$$

"as $p < q$ and $L < d$".
So,

$$y_{N+m} \leq 1.$$

Now, we will use induction hypothesis and the monotonicity properties of the function y_{N+m}, to show that $y_{N+m} \geq \frac{p+L}{q+d}$.
So we will substitute $y_{N+m-(k+1)} = 1$ and $y_{N+m-1} = \frac{p+L}{q+d}$ in the following function,

$$y_{N+m} = \frac{p+y_{N+m-1}+Ly_{N+m-(k+1)}}{q+y_{N+m-1}+dy_{N+m-(k+1)}}$$

since y_{N+m} is increasing in y_{N+m-1} and decreasing in $y_{N+m-(k+1)}$, we get the following,

$$y_{N+m} = \frac{p + y_{N+m-1} + Ly_{N+m-(k+1)}}{q + y_{N+m-1} + dy_{N+m-(k+1)}} \geq \frac{p + \frac{p+L}{q+d} + L(1)}{q + \frac{p+L}{q+d} + d(1)} = \frac{(p+L)\left[1 + \frac{1}{q+d}\right]}{(q+d)\left[1 + \frac{1}{q+d}\frac{p+L}{q+d}\right]}$$

but $\frac{p+L}{q+d} < 1.$

So,

$$\frac{(p+L)\left[1 + \frac{1}{q+d}\right]}{(q+d)\left[1 + \frac{1}{q+d}\frac{p+L}{q+d}\right]} > \frac{p+L}{q+d}\left[\frac{1 + \frac{1}{q+d}}{1 + \frac{1}{q+d}}\right] = \frac{p+L}{q+d}$$

then

$$y_{N+m} \geq \frac{p+L}{q+d}$$

So,

$$y_{N+m} \in \left[\frac{p+L}{q+d}, 1\right]$$

2. Assume that $L > d$, $p > q$, $dp > Lq$ and $\mid Lq - dp \mid > \mid x(L-d) \mid$ then by using partial derivative we can show that $f(x, y)$ is decreasing in both arguments.

Now, for some $N > 0$, and $1 \leq y_{N-k}, \ldots, y_{N-1}, y_N \leq \frac{p+L}{q+d}$

we have the following result as $p > q$ and $L > d$

$$y_{N+1} = \frac{p + y_N + Ly_{N-k}}{q + y_N + dy_{N-k}} \geq \frac{q + y_N + Ly_{N-k}}{q + y_N + dy_{N-k}} \geq \frac{q + y_N + dy_{N-k}}{q + y_N + dy_{N-k}} = 1.$$

So,

$$y_{N+1} \geq 1.$$

Also,

$$y_{N+1} = \frac{p + y_N + Ly_{N-k}}{q + y_N + dy_{N-k}}$$

since y_{N+1} is decreasing in y_{N-k} for each fixed y_N, then by substituting $y_{N-k} = 1$, in the previous function we get the following,

$$y_{N+1} = \frac{p + y_N + Ly_{N-k}}{q + y_N + dy_{N-k}} \leq \frac{p + y_N + L(1)}{q + y_N + d(1)} = \frac{(p+L)\left[1 + \frac{y_N}{p+L}\right]}{(q+d)\left[1 + \frac{y_N}{q+d}\right]}$$

but $\frac{1}{p+L} < \frac{1}{q+d}$.

So,

$$\frac{(p+L)\left[1 + \frac{y_N}{p+L}\right]}{(q+d)\left[1 + \frac{y_N}{q+d}\right]} < \frac{p+L}{q+d}\left[\frac{1 + \frac{y_N}{q+d}}{1 + \frac{y_N}{q+d}}\right] = \frac{p+L}{q+d}$$

then

$$y_{N+1} \leq \frac{p+L}{q+d}.$$

By Mathematical Induction we can see that $y_n \in \left[1, \frac{p+L}{q+d}\right]$, for all $n > N$. We proved that $y_{N+1} \in \left[1, \frac{p+L}{q+d}\right]$, so we just will show that if $y_{N+m-1} \in \left[1, \frac{p+L}{q+d}\right]$ then $y_{N+m} \in \left[1, \frac{p+L}{q+d}\right]$.

$$y_{N+m} = \frac{p + y_{N+m-1} + Ly_{N+m-(k+1)}}{q + y_{N+m-1} + dy_{N+m-(k+1)}} \geq \frac{q + y_{N+m-1} + Ly_{N+m-(k+1)}}{q + y_{N+m-1} + dy_{N+m-(k+1)}}$$

also,

$$\frac{q + y_{N+m-1} + Ly_{N+m-(k+1)}}{q + y_{N+m-1} + dy_{N+m-(k+1)}} \geq \frac{q + y_{N+m-1} + dy_{N+m-(k+1)}}{q + y_{N+m-1} + dy_{N+m-(k+1)}} = 1$$

"since $p > q$ and $L > d$".
So,

$$y_{N+m} \geq 1.$$

Also,
we will use induction hypothesis and the monotonicity properties of the function y_{N+m}, to show that $y_{N+m} \leq \frac{p+L}{q+d}$.
Since y_{N+m} is decreasing in $y_{N+m-(k+1)}$ for each fixed y_{N+m-1}, then by substituting $y_{N+m-(k+1)} = 1$, in the previous function we get the following,

$$y_{N+m} = \frac{p + y_{N+m-1} + Ly_{N+m-(k+1)}}{q + y_{N+m-1} + dy_{N+m-(k+1)}} \leq \frac{p + y_{N+m-1} + L(1)}{q + y_{N+m-1} + d(1)} = \frac{(p+L)\left[1 + \frac{y_{N+m-1}}{p+L}\right]}{(q+d)\left[1 + \frac{y_{N+m-1}}{q+d}\right]}$$

but $\frac{1}{p+L} < \frac{1}{q+d}$.

So,

$$\frac{(p+L)\left[1 + \frac{y_{N+m-1}}{p+L}\right]}{(q+d)\left[1 + \frac{y_{N+m-1}}{q+d}\right]} < \frac{p+L}{q+d}\left[\frac{1 + \frac{y_{N+m-1}}{q+d}}{1 + \frac{y_{N+m-1}}{q+d}}\right] = \frac{p+L}{q+d}$$

then

$$y_{N+m} \leq \frac{p+L}{q+d}.$$

The proof is complete.

□

3.3 Analysis Of Semi-Cycles

Our aim in this section is to study the semi-cycles behavior of solutions of Eq. (3.2.1) relative to the equilibrium point $\bar{y}$ and relative to the end points of the invariant interval of Eq. (3.2.1).

Now we give the definitions for the positive and negative semi-cycle of the solution of Eq. (3.2.1), relative to an equilibrium point $\bar{y}$.

Definition 3.3.1. *[19] A positive semi-cycle of the solution $\{y_n\}$ of Eq.* (3.2.1) *consists of a " string " of terms $\{y_l, y_{l+1}, \ldots, y_m\}$, all greater than or equal to the equilibrium point $\bar{y}$, with $l \geq -k$ and $m \leq \infty$ and such that,*

$$\textit{either } \; l = -k \; \textit{ or } \; l > -k \; \textit{ and } \; y_{l-1} < \bar{y}$$

and

$$\textit{either } \; m = \infty \; \textit{ or } \; m < \infty \; \textit{ and } \; y_{m+1} < \bar{y}.$$

Definition 3.3.2. *[19] A negative semi-cycle of the solution $\{y_n\}$ of Eq.* (3.2.1) *consists of a " string " of terms $\{y_l, y_{l+1}, \ldots, y_m\}$ all less than or equal to the equilibrium point $\bar{y}$, with $l \geq -k$ and $m \leq \infty$ and such that*

$$\textit{either } \; l = -k \; \textit{ or } \; l > -k \; \textit{ and } \; y_{l-1} \geq \bar{y}$$

and

$$\text{either } m = \infty \text{ or } m < \infty \text{ and } y_{m+1} \geq \bar{y}.$$

The first semi-cycle of a solution starts with the term y_{-k} it is positive if $y_{-k} \geq \bar{y}$ and negative if $y_{-k} < \bar{y}$

Definition 3.3.3. *[20] A solution $\{y_n\}$ of Eq.* (3.2.1) *is called non-oscillatory if there exists $N \geq -k$ such that $y_n > \bar{y}$ for all $n \geq N$ or $y_n < \bar{y}$ for all $n \leq N$.*
and a solution $\{y_n\}$ is called oscillatory if it is not non-oscillatory.

Definition 3.3.4. *[9]*

1. *A solution $\{y_n\}_{n=-k}^{\infty}$ of a difference equation is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.*

2. *A solution $\{y_n\}_{n=-k}^{\infty}$ of a difference equation is said to be periodic with prime period p or a p-cycle if it is periodic with period p and p is the least positive integer for which $x_{n+p} = x_n$.*

Definition 3.3.5. *Let* $\{y_n\}_{n=-k}^{\infty}$ *be a solution of Eq.* (3.2.1)*, we say that the solution has a prime period two if the solution eventually takes the form:*

$$\ldots, \phi, \psi, \phi, \psi, \ldots$$

where ϕ *and* ψ *are distinct and positive.*

Theorem 3.3.1. *If* k *is even, then Eq.* (3.2.1) *has no nonnegative prime period two solution.*

Proof. Assume for the sake of contradiction that there exist distinct positive real numbers ϕ and ψ, such that

$$\ldots, \phi, \psi, \phi, \psi, \ldots$$

is a prime period two solution of Eq. (3.2.1).
As k is even, so $y_n = y_{n-k}$

now, ϕ and ψ satisfy the systems

$$\phi = \frac{p + \psi + L\psi}{q + \psi + d\psi}$$

and

$$\psi = \frac{p + \phi + L\phi}{q + \phi + d\phi}.$$

So,

$$\phi q + \phi\psi + d\phi\psi = p + \psi + L\psi \tag{3.3.1}$$

$$\psi q + \phi\psi + d\phi\psi = p + \phi + L\phi. \tag{3.3.2}$$

By subtracting Eq. (3.3.2) from Eq. (3.3.1), we get

$$q(\phi - \psi) = (\psi - \phi) + L(\psi - \phi)$$

hence

$$(\psi - \phi)\,[q + L + 1] = 0.$$

As $q + L + 1 \neq 0$, then $\psi - \phi = 0 \Longrightarrow \psi = \phi$

which contradicts the hypothesis of $\phi \neq \psi$.

□

Theorem 3.3.2. *If k is odd then we have following results:*

1- The Eq. (3.2.1) *has no nonnegative prime period two in these two cases:*

- $L < 1 + q$
- $d > 1$

2- If $L > 1 + q$ and $d < 1$, then Eq. (3.2.1) *has a prime period two solution*

$$\dots, \phi, \psi, \phi, \psi, \dots$$

where the values ψ and ϕ are the solutions of the quadratic equation

$$t^2 - (\phi + \psi)t + \phi\psi = 0$$

Proof. 1- Assume for the sake of contradiction that there exist distinct and positive real numbers ϕ and ψ such that

$$\dots, \phi, \psi, \phi, \psi, \dots$$

is a prime period two solution of Eq. (3.2.1),

- k is odd then $y_{n-k} = y_{n+1}$ and in this case ϕ and ψ satisfy the following systems

$$\phi = \frac{p + \psi + L\phi}{q + \psi + d\phi}$$

and

$$\psi = \frac{p + \phi + L\psi}{q + \phi + d\psi}.$$

So,

$$q\phi + \phi\psi + d\phi^2 = p + \psi + L\phi \tag{3.3.3}$$

$$q\psi + \phi\psi + d\psi^2 = p + \phi + L\psi \tag{3.3.4}$$

subtract Eq. (3.3.4) from Eq. (3.3.3), we have

$$q(\phi - \psi) + d(\phi^2 - \psi^2) = (\psi - \phi) + L(\phi - \psi)$$

$$\Longrightarrow (\phi - \psi)\left[q + d(\phi + \psi)\right] = (\phi - \psi)\left[-1 + L\right]$$

so,

$$(\phi + \psi) = \frac{L - 1 - q}{d} = \frac{L - (1 + q)}{d} \tag{3.3.5}$$

when $L < 1 + q$ then $\phi + \psi < 0$ and this contradicts the assumption that ϕ and ψ are positive distinct real numbers.

- k is odd, from the previous steps we have,

$$\phi q + \phi\psi + d\phi^2 = p + \psi + L\phi \tag{3.3.6}$$

$$\psi q + \phi\psi + d\psi^2 = p + \phi + L\psi \tag{3.3.7}$$

By adding Eq. (3.3.6) and Eq. (3.3.7) we get,

$$\begin{aligned} q(\phi+\psi) + 2\phi\psi + d(\phi^2+\psi^2) &= 2p + (\phi+\psi) + L(\phi+\psi) \\ q(\phi+\psi) + 2\phi\psi + d(\phi^2 + 2\phi\psi - 2\phi\psi + \psi^2) &= 2p + (\phi+\psi)[1+L] \\ q(\phi+\psi) + \phi\psi(2-2d) + d(\phi+\psi)^2 &= 2p + (\phi+\psi)[1+L] \end{aligned}$$

hence

$$\phi\psi(2-2d) = 2p + (\phi+\psi)[1+L] - d(\phi+\psi)^2 - q(\phi+\psi)$$

$$= 2p + (\phi+\psi)[(1+L) - d(\phi+\psi) - q]$$

but $\phi+\psi = \frac{L-1-q}{d}$, substitute the value of $(\phi+\psi)$ in the last equation

$$\phi\psi(2-2d) = 2p + \left(\frac{L-1-q}{d}\right)\left[(1+L) - d\left(\frac{L-1-q}{d}\right) - q\right]$$

then

$$\phi\psi(2-2d) = 2p + 2\left(\frac{L-1-q}{d}\right)$$

so

$$\phi\psi = \frac{[pd + (L-1-q)]}{d(1-d)} \tag{3.3.8}$$

when $d > 1$ then $\phi\psi < 0$, this contradicts the assumption that ϕ and ψ are distinct and positive real numbers.

2- If $L > (1+q)$ and $d < 1$, then it is clear from Eq. (3.3.8) and Eq. (3.3.5) that ϕ and ψ are two distinct real roots of the quadratic equation

$$t^2 - \left(\frac{L-1-q}{d}\right)t + \frac{[pd + (L-1-q)]}{d(1-d)} = 0$$

which have the following values

$$\psi = \frac{1}{2}\left(\frac{L-1-q}{d}\right) - \frac{1}{2}\sqrt{\left(\frac{L-1-q}{d}\right)^2 - 4\left(\frac{[pd+(L-1-q)]}{d(1-d)}\right)}$$

and

$$\phi = \frac{1}{2}\left(\frac{L-1-q}{d}\right) + \frac{1}{2}\sqrt{\left(\frac{L-1-q}{d}\right)^2 - 4\left(\frac{[pd+(L-1-q)]}{d(1-d)}\right)}$$

The proof is complete.

□

Theorem 3.3.3. *[5] Assume that $f \in \mathcal{C}[(0,\infty)\times(0,\infty),(0,\infty)]$ such that $f(x,y)$is increasing in x for each fixed y, and decreasing in y for each fixed x.*

Let $\bar{y}$ be a positive equilibrium of Eq. (3.2.11)*, then every oscillatory solution of Eq.* (3.2.11) *has semi-cycles of length at least k.*

Proof. The proof of this theorem will follow by using Mathematical Induction.
When $k = 1$, then the proof of this result is presented in Theorem 1.7.4 in [16], so we just show that if the theorem is true for $k = m-1$ then it will be true when $k = m$.

Assume that $\{y_n\}$ is an oscillatory solution with $m+1$ consecutive terms $y_{N-1}, y_N, \ldots, y_{N+m-1}$ such that y_{N-1} belong to the negative semi-cycle, and the following terms belong to the positive semi-cycles.
So

$$y_{N-1} \le \bar{y} < y_{N+m-1}.$$

From the previous assumption we can conclude that, when $k = m-1$ then every oscillatory solution of Eq. (3.2.11) has semi-cycles of length at least $m-1$ terms in the positive semi-cycles.
Now by using the monotonicity properties of the function f and the induction hypothesis we obtain

$$y_{N+m} = f(y_{N+m-1}, y_{N-1}) > f(\bar{y},\bar{y}) = \bar{y}.$$

Which shows that it has at least m terms in the positive semi-cycle.
Which completes the proof. □

Theorem 3.3.4. *[16] Assume that $f \in \mathcal{C}[(0,\infty)\times(0,\infty),(0,\infty)]$ is such that $f(x,y)$is decreasing in x for each fixed y, and increasing in y for each fixed x.*
Let $\bar{y}$ be a positive equilibrium of Eq. (3.2.11)*, then except possibly for the first semi-cycle every oscillatory solution of Eq.* (3.2.11) *has semi-cycles of length k.*

Proof. In this proof we will use mathematical induction.
When $k = 1$, then the proof of this result is presented in Theorem 1.7.1 in

[16], assume that if the theorem is true when $k = m - 1$ then we will show that is true for $k = m$.

Assume that $\{y_n\}$ is a solution of equation (3.2.11) with $m + 1$ consecutive terms $y_{N-1}, y_N, \ldots, y_{N+m-1}$ such that y_{N-1} belong to the negative semi-cycle, and the following terms belong to the positive semi-cycles.
So

$$y_{N-1} \leq \bar{y} < y_{N+m-1}.$$

From the previous assumption we can conclude that, when $k = m - 1$ then every oscillatory solution of Eq. (3.2.11) has semi-cycles of length $m-1$ terms in the positive semi-cycles.
Then by using the monotonicity properties of the function f and the induction hypothesis we have

$$y_{N+m} = f(y_{N+m-1}, y_{N-1}) < f(\bar{y}, \bar{y}) = \bar{y}$$

and

$$y_{N+m+1} = f(y_{N+m}, y_N) > f(\bar{y}, \bar{y}) = \bar{y}.$$

Thus

$$y_{N+m} < \bar{y} < y_{N+m+1}$$

Which shows that it has m terms in the positive semi-cycle, which completes the proof.

□

Theorem 3.3.5. *[16] Assume that $f \in \mathcal{C}[(0,\infty)\times(0,\infty),(0,\infty)]$ and that $f(x,y)$ is increasing in both arguments.*
Let $\bar{y}$ be a positive equilibrium of Eq. (3.2.11). *Then except possibly for the first semi-cycle, every oscillatory solution of Eq.* (3.2.11) *has semi-cycles of length k.*

Proof. We will use mathematical induction to proof this theorem.
When $k = 1$, then the proof of this result is presented in Theorem 1.7.3 in [16], assume that is true for $k = m-1$, then we will prove the theorem when $k = m$.

Assume that $\{y_n\}$ is an oscillatory solution with $m+1$ consecutive terms $y_{N-1}, y_N, \dots, y_{N+m-1}$ in a positive semi-cycle

$$y_{N-1} \geq \bar{y},\ y_N \geq \bar{y},\ \dots,\ y_{N+m-1} > \bar{y}$$

with at least half of the inequalities being strict. From the previous assumption we can conclude that, when $k = m-1$ then every oscillatory solution of Eq. (3.2.11) has semi-cycles of length $m-1$.
Then by using the increasing character of f and the induction hypothesis we obtain:

$$y_{N+m} = f(y_{N+m-1}, y_{N-1}) > f(\bar{y},\bar{y}) = \bar{y}$$

So it followed by induction that all the terms of this solution belong to this positive semi-cycle, which is a contradiction.

□

Theorem 3.3.6. *[5] Assume that $f \in \mathcal{C}[(0,\infty)\times(0,\infty),(0,\infty)]$and that $f(x,y)$ is decreasing in both arguments.*
Let $\bar{y}$ be a positive equilibrium of Eq. (3.2.11)*, then every oscillatory solution of Eq.* (3.2.11) *has semi-cycles of length at most k.*

Proof. When $k = 1$, then the proof of this result is presented in Theorem 1.7.2 in [16], assume the theorem holds for $k = m-1$ then by using mathematical induction we can prove the theorem for the case $k = m$.

Assume that $\{y_n\}$ is an oscillatory solution with $m+1$ consecutive terms $y_{N-1}, y_N, \ldots, y_{N+m-1}$ in a positive semi cycles.
$y_{N-1} \geq \bar{y}$, $y_N \geq \bar{y}$, $\ldots$, $y_{N+m-1} > \bar{y}$, with at least half of the inequality begin strict. We can conclude from the previous assumption that, when $k = m-1$ then every oscillatory solution of Eq. (3.2.11) has semi-cycles of length at most $m-1$ terms in the positive semi-cycles.
Then by using the decreasing character of f and the induction hypothesis we obtain,

$$y_{N+m} = f(y_{N+m-1}, y_{N-1}) < f(\bar{y}, \bar{y}) = \bar{y}$$

which completes the proof.

□

Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq. (3.2.1) then the following are true:

$$y_{n+1} - 1 = (d-L)\left[\frac{(\frac{p-q}{d-L}) - y_{n-k}}{q + y_n + dy_{n-k}}\right] \tag{3.3.9}$$

Notice that $\frac{p-q}{d-L} < 0$, thus $\frac{p-q}{d-L} < \frac{p+L}{d+q}$ so we have the following equation:

$$y_{n+1} - 1 = (d-L)\left[\frac{(\frac{p-q}{d-L}) - y_{n-k}}{q + y_n + dy_{n-k}}\right] < (d-L)\left[\frac{(\frac{p+L}{d+q}) - y_{n-k}}{q + y_n + dy_{n-k}}\right]. \tag{3.3.10}$$

Also,

$$\left(y_{n+1} - \frac{p+L}{q+d}\right) = \frac{(1 - \frac{p+L}{q+d})y_n + (p+q)[1 - y_{n-k}]}{q + y_n + dy_{n-k}} \tag{3.3.11}$$

Case I:

We will analyze the semi-cycles of the solution $\{y_n\}_{n=-k}^{\infty}$ under the assumption that

$$p < q, \; L < d \;\; and \;\; dp > Lq. \tag{3.3.12}$$

By using Eqs. (3.3.9), (3.3.10) and (3.3.11) we get the following results:

Lemma 3.3.1. *Assume that* (3.3.12) *holds, and let* $\{y_n\}_{n=-k}^{\infty}$ *be a solution of Eq.* (3.2.1)*, then the following statements are true:*

1. *For some* $N \geq 0$*, if* $y_{N-k} \geq \frac{p+L}{q+d}$*, then* $y_{N+1} \leq 1$.
2. *For some* $N \geq 0$*, if* $y_{N-k} < \frac{p+L}{q+d}$*, then* $y_{N+1} > 1$.
3. *For some* $N \geq 0$*, if* $y_{N-k} \leq 1$*, then* $y_{N+1} \geq \frac{p+L}{q+d}$.
4. *For some* $N \geq 0$*, if* $\frac{p+L}{q+d} \leq y_{N-k} \leq 1$*, then* $\frac{p+L}{q+d} \leq y_{N+1} \leq 1$.
5. *For some* $N \geq 0$*, if* $\frac{p+L}{q+d} \leq y_{N-k}, \ldots, y_{N-1}, y_N \leq 1$,

   then $y_n \in \left[\frac{p+L}{q+d}, 1\right]$ *for* $n \geq N$*, where* $\left[\frac{p+L}{q+d}, 1\right]$ *is an invariant interval of Eq.* (3.1.2).
6. $\frac{p+L}{q+d} < \bar{y} < 1$.

Proof. Assume that Eq. (3.3.12) holds, then

1. for some $N \geq 0$ if $y_{N-k} \geq \frac{p+L}{q+d}$, then we can conclude that $y_{N+1} - 1 \leq 0$ by using Eq. (3.3.10). So $y_{N+1} \leq 1$.
2. for some $N \geq 0$ and $y_{N-k} < \frac{p+L}{q+d}$, when $y_{N-k} < \frac{p-q}{d-L}$ then $y_{N+1} - 1 > 0$ but $y_{N-k} < \frac{p-q}{d-L} < \frac{p+L}{d+q}$ then by using Eq. (3.3.10) we can conclude that $y_{N+1} - 1 > 0$ and so $y_{N+1} > 1$.
3. for some $N \geq 0$ if $y_{N-k} \leq 1$ then from Eq. (3.3.11) we can conclude that $y_{N+1} - \frac{p+L}{q+d} \geq 0$, so $y_{N+1} \geq \frac{p+L}{q+d}$.
4. for some $N \geq 0$, $\frac{p+L}{q+d} \leq y_{N-k} \leq 1$, we see from (1) that if $y_{N-k} \geq \frac{p+L}{q+d}$ then $y_{N+1} \leq 1$, also we see that if $y_{N-k} \leq 1$ then $y_{N+1} \geq \frac{p+L}{q+d}$ so we conclude that $\frac{p+L}{q+d} \leq y_{N+1} \leq 1$.

5. if for some $N \geq 0$, then we see from (4) that if $\frac{p+L}{q+d} \leq y_{N-k} \leq 1$ then $\frac{p+L}{q+d} \leq y_{N+1} \leq 1$. Also we can see that if $\frac{p+L}{q+d} \leq y_{N-k}, \dots, y_{N-1}, y_N \leq 1$, then $y_n \in \left[\frac{p+L}{q+d}, 1\right]$ for $n \geq N$ by using Eqs. (3.3.10) and (3.3.11), so $\left[\frac{p+L}{q+d}, 1\right]$ is an invariant interval for Eq. (3.1.2).

6. By using (5), as $\left[\frac{P+L}{q+d}, 1\right]$ is an invariant interval, then $\frac{p+L}{q+d} < \bar{y} < 1$.

□

Theorem 3.3.7. *Assume that Eq.* (3.3.12) *holds. Then every non trivial and oscillatory solution of Eq.* (3.2.1) *which lies in the interval* $\left[\frac{p+L}{q+d}, 1\right]$ *oscillates about* $\bar{y}$ *with semi-cycles of length at least* k.

Proof. Assume that Eq. (3.3.12) holds then Eq. (3.2.1) is increasing in x and decreasing in y, $\forall x, y \in \left[\frac{p+L}{q+d}, 1\right]$ so we see by using Theorem (3.3.3) that every non trivial and oscillatory solution of Eq. (3.2.1) has semi-cycle of length at least k. □

Case II:

Now, we will analyze the semi-cycles of the solution $\{y_n\}_{n=-k}^{\infty}$ under the assumption that

$$p > q, \; L > d \text{ and } dp > Lq. \tag{3.3.13}$$

The following results is a direct consequences of Eqs. (3.3.9), (3.3.10) and (3.3.11)

Lemma 3.3.2. *Assume that* (3.3.13) *holds, and let* $\{y_n\}_{n=-k}^{\infty}$ *be a solution of Eq.* (3.2.1)*, then the following statements are true:*

1. *For some* $N \geq 0$*, if* $y_{N-k} \leq \frac{p+L}{q+d}$*, then* $y_{N+1} \geq 1$.

2. *For some* $N \geq 0$*, if* $y_{N-k} \geq \frac{p+L}{q+d}$*, then* $y_{N+1} \leq 1$.

3. For some $N \geq 0$*, if* $y_{N-k} \geq 1$*, then* $y_{N+1} \leq \frac{p+L}{q+d}$.

4. For some $N \geq 0$*, if* $1 \leq y_{N-k} \leq \frac{p+L}{q+d}$*, then* $1 \leq y_{N+1} \leq \frac{p+L}{q+d}$.

5. For some $N \geq 0$*, if* $1 \leq y_{N-k}, \ldots, y_{N-1}, y_N \leq \frac{p+L}{q+d}$ *,*

then $y_n \in \left[1, \frac{p+L}{q+d}\right]$ *for* $n \geq N$*. where* $\left[1, \frac{p+L}{q+d}\right]$ *is an invariant interval of Eq.* (3.1.2).

6. $1 < \bar{y} < \frac{p+L}{q+d}$.

Proof. Assume that Eq. (3.3.13) holds, then

1. for some $N \geq 0$, when $y_{N-k} \leq \frac{p-q}{d-L}$ then $y_{N+1} - 1 \geq 0$ but $y_{N-k} \leq \frac{p-q}{d-L} < \frac{p+L}{d+q}$ then by using Eq. (3.3.10) we can conclude that $y_{N+1} - 1 \geq 0$ and so $y_{N+1} \geq 1$.
2. for some $N \geq 0$ if $y_{N-k} \geq \frac{p+L}{q+d}$, then we can conclude that $y_{N+1} - 1 \leq 0$ by using Eq. (3.3.10). So $y_{N+1} \leq 1$.
3. for some $N \geq 0$ if $y_{N-k} \geq 1$ then from Eq. (3.3.11) we can conclude that $y_{N+1} - \frac{p+L}{q+d} \leq 0$, so $y_{N+1} \leq \frac{p+L}{q+d}$.
4. for some $N \geq 0$, $1 \leq y_{N-k} \leq \frac{p+L}{q+d}$, we see from (1) that if $y_{N-k} \leq \frac{p+L}{q+d}$ then $y_{N+1} \geq 1$, also we see that if $y_{N-k} \geq 1$ then $y_{N+1} \leq \frac{p+L}{q+d}$ so we conclude that $1 \leq y_{N+1} \leq \frac{p+L}{q+d}$.
5. if for some $N \geq 0$, then we see from (4) that if $1 \leq y_{N-k} \leq \frac{p+L}{q+d}$ then $1 \leq y_{N+1} \leq \frac{p+L}{q+d}$. Also we can see that if $1 \leq y_{N-k}, \ldots, y_{N-1}, y_N \leq \frac{p+L}{q+d}$, then $y_n \in \left[1, \frac{p+L}{q+d}\right]$ for $n \geq N$ by using Eqs. (3.3.10) and (3.3.11), so $\left[1, \frac{p+L}{q+d}\right]$ is an invariant interval for Eq. (3.1.2).
6. By using (5), as $\left[1, \frac{P+L}{q+d}\right]$ is an invariant interval, then $1 < \bar{y} < \frac{p+L}{q+d}$.

□

Theorem 3.3.8. *Assume that Eq.* (3.3.13) *holds. Then every non trivial and oscillatory solution of Eq.* (3.2.1) *which lies in the interval* $\left[1, \frac{p+L}{q+d}\right]$ *oscillates about* $\bar{y}$ *with semi-cycles of length at most k.*

Proof. Assume that Eq. (3.3.13) holds then Eq. (3.2.1) is decreasing in both arguments, $\forall x, y \in \left[1, \frac{p+L}{q+d}\right]$ so we see by using Theorem (3.3.6) that every non trivial and oscillatory solution of Eq. (3.2.1) has semi-cycles of length at most k. □

<u>Case III:</u>

We will analyze the semi-cycles of the solution $\{y_n\}_{n=-k}^{\infty}$ under assumption that

$$p = q \text{ and } d = L. \tag{3.3.14}$$

In this case Eq. (3.3.11) reduces to

$$y_{n+1} - 1 = \frac{(p+q)\,[1 - y_{n-k}]}{q + y_n + dy_{n-k}} \tag{3.3.15}$$

so, the following results follow directly:

Lemma 3.3.3. *Let* $\{y_n\}_{n=-k}^{\infty}$ *be a solution of Eq.* (3.2.1), *and assume that* (3.3.14) *holds, then the following statements are true:*

1. *For some* $N \geq 0, \quad y_{N-k} < 1,$ *then* $y_{N+1} > 1.$
2. *For some* $N \geq 0, \quad y_{N-k} = 1,$ *then* $y_{N+1} = 1.$
3. *For some* $N \geq 0, \quad y_{N-k} > 1,$ *then* $y_{N+1} < 1.$

Proof. Assume that Eq. (3.3.14) holds, then

1. for some $N \geq 0$ if $y_{N-k} < 1$ then we conclude that $y_{N+1} - 1 > 0$ and so $y_{N+1} > 1$ by using Eq. (3.3.15).

2. for some $N \geq 0$ if $y_{N-k} = 1$ then we get $y_{N+1} - 1 = 0$ from Eq. (3.3.15). So $y_{N+1} = 1$

3. for some $N \geq 0$, if $y_{N-k} > 1$, then $y_{N+1} - 1 < 0$, which implies $y_{N+1} < 1$

□

Corollary 3.3.1. *Assume that Eq.* (3.3.14) *holds. Then every non trivial solution of Eq.* (3.2.1) *oscillates about the equilibrium point* $\bar{y}$*.*

Proof. We notice that by using lemma (3.3.3) if $y_{N-k} \leq 1$, then $y_{N+1} \geq 1$, also if $y_{N-k} \geq$ then $y_{N+1} \leq 1$, which means that the solution $\{y_n\}_{n=-k}^{\infty}$ oscillates about the equilibrium point $\bar{y}$= 1. □

3.4 Global Stability

In this section we consider the global asymptotic stability of Eq. (3.2.1).
In section (3.2), we investigated local stability of the positive equilibrium point so it is sufficient to investigate the globally attractive of positive equilibrium point.

Now, we present some theorems which will be used in this section.

Theorem 3.4.1. *[22] [16] Let $I = [a, b]$ be some interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \longrightarrow [a, b]$$

is a continuous function satisfying the following properties:

(a) $f(x, y)$ is non decreasing in x, and non increasing in y where $x, y \in [a, b]$.

(b) If $(m, \mu) \in [a, b] \times [a, b]$ is a solution of the system.

$$m = f(m, \mu) \text{ and } \mu = f(\mu, m),$$

then $m = \mu$.
Then Eq. (3.2.11) has a unique equilibrium point $\bar{y}$ and every solution of Eq. (3.2.11) converges to $\bar{y}$.

Proof. Set

$$m_0 = a \text{ and } \mu_0 = b$$

and for $i = 1, 2, \ldots$ set

$$\mu_i = f(\mu_{i-1}, m_{i-1}) \text{ and } m_i = f(m_{i-1}, \mu_{i-1}).$$

Now observe that for each $i \geq 0$,

$$m_0 \leq m_1 \leq \cdots \leq m_i \leq \cdots \leq \mu_i \leq \cdots \leq \mu_1 \leq \mu_0$$

and

$$m_i \leq y_k \leq \mu_i \text{ for } k \geq 2i+1.$$

Set

$$m = \lim_{i \longrightarrow \infty} m_i \quad \text{and} \quad \mu = \lim_{i \longrightarrow \infty} \mu_i.$$

Then

$$\mu \geq \limsup_{i \longrightarrow \infty} y_i \geq \liminf_{i \longrightarrow \infty} y_i \geq m$$

and by the continuity of f,

$$m = f(m, \mu) \text{ and } \mu = f(\mu, m).$$

In view of (b),

$$\mu = m$$

from which the result follows.

□

Theorem 3.4.2. *[16] Let $I = [a, b]$ be an interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \longrightarrow [a, b]$$

is a continuous function satisfying the following properties:

(a) $f(x, y)$ is non increasing in each of its arguments.

(b) If $(m, \mu) \in [a, b] \times [a, b]$ is a solution of the system

$$\mu = f(m, m) \text{ and } m = f(\mu, \mu),$$

then $m = \mu$.
Then Eq. (3.2.11) has a unique equilibrium point $\bar{y}$ and every solution of Eq. (3.2.11) converges to $\bar{y}$.

Proof. Set

$$m_0 = a \text{ and } \mu_0 = b$$

and for $i = 1, 2, \ldots$ set

$$\mu_i = f(m_{i-1}, m_{i-1}) \text{ and } m_i = f(\mu_{i-1}, \mu_{i-1}).$$

Now observe that for each $i \geq 0$,

$$m_0 \leq m_1 \leq \cdots \leq m_i \leq \cdots \leq \mu_i \leq \ldots \mu_1 \leq \mu_0$$

and

$$m_i \leq y_k \leq \mu_i \quad \text{for } k \geq 2i + 1.$$

Set

$$m = \lim_{i \longrightarrow \infty} m_i \quad \text{and} \quad \mu = \lim_{i \longrightarrow \infty} \mu_i.$$

Then clearly

$$\mu \geq \limsup_{i \longrightarrow \infty} y_i \geq \liminf_{i \longrightarrow \infty} y_i \geq m$$

and by the continuity of f,

$$\mu = f(m, m) \text{ and } m = f(\mu, \mu).$$

In view of (b),

$$\mu = m = \bar{y}$$

from which the result follows.

□

Theorem 3.4.3. *[22] [9] Let $I = [a, b]$ be an interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \longrightarrow [a, b]$$

is a continuous function satisfying the following properties:

(a) $f(x, y)$ is non increasing in x for each fixed y and $f(x, y)$ is non decreasing in y for each fixed x, where $x, y \in [a, b]$.

(b) The difference Eq. (3.2.11) *has no solutions of prime period two in [a,b]. Then the difference Eq.* (3.2.11) *has a unique equilibrium point $\bar{y} \in [a, b]$ and every solution of it converges to $\bar{y}$.*

Proof. Set

$$m_0 = a \ \ and \ \ \mu_0 = b$$

and for $i = 1, 2, \dots$ set

$$\mu_i = f(m_{i-1}, \mu_{i-1}) \ \ and \ \ m_i = f(\mu_{i-1}, m_{i-1}).$$

Now observe that for each $i \geq 0$,

$$m_0 \leq m_1 \leq \cdots \leq m_i \leq \cdots \leq \mu_i \leq \cdots \leq \mu_1 \leq \mu_0,$$

and

$$m_i \leq y_k \leq \mu_i \ \ for \ \ k \geq 2i + 1.$$

Set

$$m = \lim_{i \longrightarrow \infty} m_i \ \ and \ \ \mu = \lim_{i \longrightarrow \infty} \mu_i.$$

Then clearly

$$\mu \geq \limsup_{i \longrightarrow \infty} y_i \geq \liminf_{i \longrightarrow \infty} y_i \geq m$$

and by the continuity of f,

$$\mu = f(m, \mu) \ \ and \ \ m = f(\mu, m).$$

In view of (b),

$$\mu = m = \bar{y}$$

from which the result follows.

□

Theorem 3.4.4. *[16] Let $I = [a, b]$ be an interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \longrightarrow [a, b]$$

is a continuous function satisfying the following properties:

(a) $f(x, y)$ is non decreasing in each of its arguments.

(b) If $(m, \mu) \in [a, b] \times [a, b]$ is a solution of the system

$$\mu = f(\mu, \mu) \ \ and \ \ m = f(m, m)$$

then $m = \mu$.
Then Eq. (3.2.11) *has a unique positive equilibrium point, and every positive solution of Eq.* (3.2.11) *converges to* $\bar{y}$.

Proof. Set

$$m_0 = a \text{ and } \mu_0 = b$$

and for $i = 1, 2, \dots$ set

$$\mu_i = f(\mu_{i-1}, \mu_{i-1}) \text{ and } m_i = f(m_{i-1}, m_{i-1}).$$

Now observe that for each $i \geq 0$,

$$m_0 \leq m_1 \leq \cdots \leq m_i \leq \cdots \leq \mu_i \leq \cdots \leq \mu_1 \leq \mu_0$$

and

$$m_i \leq y_k \leq \mu_i \text{ for } k \geq 2i + 1.$$

Set

$$m = \lim_{i \longrightarrow \infty} m_i \text{ and } \mu = \lim_{i \longrightarrow \infty} \mu_i.$$

Then clearly

$$\mu \geq \limsup_{i \longrightarrow \infty} y_i \geq \liminf_{i \longrightarrow \infty} y_i \geq m$$

and by the continuity of f,

$$m = f(m, m) \text{ and } \mu = f(\mu, \mu).$$

In view of (b)

$$\mu = m = \bar{y}$$

from which the result follows.

□

From the above discussion we have the main result of this section as follows:

Theorem 3.4.5. *Assume that $p > q$, $L > d$ and $dp > Lq$ then the equilibrium point of Eq.* (3.2.1) *is globally asymptotically stable in the interval* $\left[1, \frac{p+L}{q+d}\right]$.

Proof. We use Theorem (3.4.2), assume that $p > q$, $L > d$ and $dp > Lq$ and suppose that $\left[1, \frac{p+L}{q+d}\right]$ is an invariant interval for the function

$$f(x, y) = \frac{p + x + Ly}{q + x + dy}.$$

We saw that in this interval the function $f(x, y)$ is decreasing in both arguments, so part (a) of Theorem (3.4.2) holds.

Now, let $(m, \mu) \in [a, b] \times [a, b]$ is a solution of the system
$f(m, m) = \mu$ and $f(\mu, \mu) = m$.
then

$$m = \frac{p + \mu + L\mu}{q + \mu + d\mu} \qquad \textit{and} \qquad \mu = \frac{p + m + Lm}{q + m + dm}.$$

But we saw that this equation has no period two solution
" when $y_n = y_{n-k}$, k is even ".

So, the only solution is $m = \mu$.

The two conditions of Theorem (3.4.2) hold, then every solution of Eq. (3.2.1) converge to $\bar{y}$ in the interval $\left[1, \frac{p+L}{q+d}\right]$.
So the equilibrium point $\bar{y}$ is globally attractive.

□

Theorem 3.4.6. *Assume that $p < q$, $L < d$ and $dp > Lq$ then the equilibrium point of Eq.* (3.2.1) *is globally asymptotically stable in the interval* $\left[\frac{p+L}{q+d}, 1\right]$.

Proof. We use Theorem (3.4.1). Assume that $p < q$, $L < d$ and $dp > Lq$ and suppose that $\left[\frac{p+L}{q+d}, 1\right]$ is an invariant interval for the function

$$f(x,y) = \frac{p+x+Ly}{q+x+dy}.$$

We saw that in this interval the function $f(x,y)$ is increasing in x and decreasing in y, so part (a) of Theorem (3.4.1) holds.

Now, let $(m,\mu) \in [a,b] \times [a,b]$ be a solution of the system
$f(m,\mu) = m$ and $f(\mu,m) = \mu$
then

$$m = \frac{p+m+L\mu}{q+m+d\mu} \qquad \textit{and} \qquad \mu = \frac{p+\mu+Lm}{q+\mu+dm}$$

Then $m = \mu$.

So, the two conditions of Theorem (3.4.1) hold. Then by Theorem (3.4.1) every solution of Eq. (3.2.1) converge to $\bar{y}$ in the interval $\left[\frac{p+L}{q+d}, 1\right]$. So the equilibrium point $\bar{y}$ is globally attractive.

Since $\bar{y}$ is asymptotically stable, then by Definition (3.1.2), $\bar{y}$ is globally asymptotically stable.

□

4 The Special cases $\alpha\beta\gamma ABC = 0$

In this chapter we will study the character of solution of Eq. (3.1.1) where one, two or more of the parameters in Eq. (3.1.1) are zeros. There are many equations that arise by considering zero parameters.
Notice that some of these equations have been studied and few of them are meaningless such as the case when all the parameters in the denominator or the numerator are zero.

4.1 One Parameter =0

In this section we will study the character of the solution of Eq. (3.1.1) where one parameter in Eq. (3.1.1) equals zero. There are six cases, namely:

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{A + Bx_n}, \qquad n = 0, 1, 2, \ldots \tag{4.1.1}$$

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{A + Cx_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{4.1.2}$$

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{4.1.3}$$

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + Bx_n + Cx_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{4.1.4}$$

$$x_{n+1} = \frac{\alpha + \gamma x_{n-k}}{A + Bx_n + Cx_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{4.1.5}$$

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{A + Bx_n + Cx_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{4.1.6}$$

Where the remaining parameters $\alpha,\ \beta,\ \gamma,\ \textit{and}\ A, B, C$ are non-negative real numbers and the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are arbitrary real numbers, and the denominator is nonzero.

4.1.1 The Case C = 0 : $x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-k}}{A+Bx_n}$

Lemma 4.1.1. *The change of variables $x_n = \frac{A}{B}y_n$, reduces Eq. (4.1.1) to the difference equation*

$$y_{n+1} = \frac{p + qy_n + ry_{n-k}}{1 + y_n} \tag{4.1.7}$$

where $p = \frac{\alpha B}{A^2}$, $q = \frac{\beta}{A}$, and $r = \frac{\gamma}{A}$ and the initial conditions $y_{-k}, \ldots, y_0$ are arbitrary nonnegative real numbers.

Proof. Substitute $x_n = \frac{A}{B}y_n$ in Eq. (4.1.1), we get:

$$\frac{A}{B}y_{n+1} = \frac{\alpha + \frac{\beta A}{B}y_n + \frac{\gamma A}{B}y_{n-k}}{A + \frac{BA}{B}y_n}$$

then

$$y_{n+1} = \frac{A\left[\frac{\alpha B}{A^2} + \frac{\beta}{A}y_n + \frac{\gamma}{A}y_{n-k}\right]}{A\left[1 + y_n\right]}$$

set $p = \frac{\alpha B}{A^2}$, $q = \frac{\beta}{A}$, and $r = \frac{\gamma}{A}$, then we get Eq. (4.1.7).

Eq. (4.1.7) was investigated by R.M.Sebdani and M.Dehghan in [21].

□

4.1.2 The Case B = 0 : $x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-k}}{A+Cx_{n-k}}$

Lemma 4.1.2. *The change of variables $x_n = \frac{\gamma}{C}y_n$ reduces Eq. (4.1.2) to the difference equation*

$$y_{n+1} = \frac{p + Ly_n + y_{n-k}}{q + y_{n-k}} \qquad n = 0, 1, \ldots \tag{4.1.8}$$

Where $p = \frac{\alpha C}{\gamma^2}$, $q = \frac{A}{\gamma}$ and $L = \frac{\beta}{\gamma}$.

And the initial conditions $y_{-k}, \ldots, y_0$ are arbitrary nonnegative real numbers.

Proof. Substitute $x_n = \frac{\gamma}{C} y_n$ in Eq. (4.1.2) we get,

$$\frac{\gamma}{C} y_{n+1} = \frac{\alpha + \frac{\beta\gamma}{C} y_n + \frac{\gamma^2}{C} y_{n-k}}{A + \frac{C\gamma}{C} y_{n-k}}$$

so,

$$y_{n+1} = \frac{\gamma \left[\frac{\alpha C}{\gamma^2} + \frac{\beta}{\gamma} y_n + y_{n-k} \right]}{\gamma \left[\frac{A}{\gamma} + y_{n-k} \right]}$$

set $p = \frac{\alpha C}{\gamma^2}$, $q = \frac{A}{\gamma}$ and $L = \frac{\beta}{\gamma}$.

Then we get Eq. (4.1.8). □

The only positive equilibrium point of Eq. (4.1.8) is

$$\bar{y} = \frac{(L + 1 - q) + \sqrt{(q - L - 1)^2 + 4p}}{2}.$$

And the linearized equation is

$$z_{n+1} + \frac{-L(q + \bar{y})}{(q + \bar{y})^2} z_n + \frac{(p + L\bar{y}) - q}{(q + \bar{y})^2} z_{n-k} = 0.$$

Theorem 4.1.1. *Assume that $p + L > q$, where $\bar{y} > L$, $\bar{y} > 1$ and $L < 1$ then equilibrium point $\bar{y}$ of Eq.* (4.1.8) *is locally stable.*

The proof follows immediately from Theorem (3.2.2).

Theorem 4.1.2. *Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq.* (4.1.8)*, then Eq.* (4.1.8) *has no solution of prime period two in the following two cases:*

- *k is even.*
- *k is odd and $q + L > 1$*

Proof. Let

$$\ldots, \phi, \psi, \phi, \psi, \ldots$$

be a period two solution of Eq. (4.1.8), where ϕ *and* ψ are real numbers.

k is even, then we have the following systems:

$$\phi = \frac{p + \psi + L\psi}{q + \psi} \quad and \quad \psi = \frac{p + \phi + L\phi}{q + \phi}. \tag{4.1.9}$$

By simplifying (4.1.9) we obtain,

$$(\phi - \psi)\,[q + 1 + L] = 0$$

as $q \neq -(1 + L)$ then $\phi = \psi$.

- If k is odd, then we have the following systems

$$\psi = \frac{p + L\phi + \psi}{q + \psi} \quad and \quad \phi = \frac{p + L\psi + \phi}{q + \phi} \tag{4.1.10}$$

simplifying the relation in Eq. (4.1.10) to get,

$$(\phi - \psi)\,[q + (\phi + \psi) + L - 1] = 0$$

$\Longrightarrow \phi + \psi = 1 - (q + L)$ when $(L + q) > 1$, then $\phi + \psi < 0$
and this is a contradiction.

So, $\phi = \psi$

□

Theorem 4.1.3. *Assume that $p + L > q$ and $\bar{y} > 1$ then the equilibrium point $\bar{y}$ of Eq.* (4.1.8) *is globally attractive on the interval* $\left[1, \frac{p+L}{q}\right]$.

Proof. When $p + L > q$, and $x > 1$ then the function

$$f(x, y) = \frac{p + Lx + y}{q + y}$$

is decreasing in y and increasing in x, where $x, y \in \left[1, \frac{p+L}{q}\right]$.

We can easily see that the equilibrium point $\bar{y}$ is globally attractive,
by using Theorem (3.4.1).

□

Theorem 4.1.4. *Assume that $p + L > q$ and $\bar{y} > 1$ then every oscillatory solution of Eq.* (4.1.8) *has semi-cycles of length at least k.*

The proof follows from Theorem (3.3.3)

Eq. (4.1.3) was investigated by A.Farhat in [13] and by A.E.Alaweneh in [3].
Also, Eq. (4.1.4) was investigated by M.Abu Alhalawa in [2].

4.1.3 The case $\beta = 0$: $x_{n+1} = \frac{\alpha+\gamma x_{n-k}}{A+Bx_n+Cx_{n-k}}$

Lemma 4.1.3. *The change of variables $x_n = \frac{\gamma}{C} y_n$ reduces Eq.* (4.1.5) *to the difference equation*

$$y_{n+1} = \frac{p + y_{n-k}}{q + dy_n + y_{n-k}} \qquad n = 0, 1, \ldots \tag{4.1.11}$$

Where $p = \frac{\alpha C}{\gamma^2}$, $q = \frac{A}{\gamma}$ and $d = \frac{B}{C}$.

And the initial conditions $y_{-k}, \ldots, y_0$ are arbitrary nonnegative real numbers.

Proof. Substitute $x_n = \frac{\gamma}{C} y_n$ in Eq. (4.1.5) we get,

$$\frac{\gamma}{C} y_{n+1} = \frac{\alpha + \gamma \frac{\gamma}{C} y_{n-k}}{A + \frac{\beta\gamma}{C} y_n + \frac{C\gamma}{C} y_{n-k}}$$

cancel $\frac{\gamma}{C}$ from both side

$$\Longrightarrow y_{n+1} = \frac{\gamma \left[\frac{\alpha C}{\gamma^2} + y_{n-k}\right]}{\gamma \left[\frac{A}{\gamma} + \frac{\beta}{C} y_n + y_{n-k}\right]}$$

set $p = \frac{\alpha C}{\gamma^2}$, $q = \frac{A}{\gamma}$ and $d = \frac{B}{C}$ we obtain Eq. (4.1.11).

□

The only positive equilibrium point of Eq. (4.1.11) is

$$\bar{y} = \frac{(1-q) + \sqrt{(q-1)^2 + 4p(d+1)}}{2(d+1)}.$$

And the linearized equation is

$$z_{n+1} + \frac{d(p+\bar{y})}{(q+d\bar{y}+\bar{y})^2} z_n + \frac{p-q-d\bar{y}}{(q+d\bar{y}+\bar{y})^2} z_{n-k} = 0.$$

Theorem 4.1.5. *The equilibrium point*

$$\bar{y} = \frac{(1-q) + \sqrt{(q-1)^2 + 4p(d+1)}}{2(d+1)}$$

is locally stable when $p > q + d$, $\bar{y} \leq 1$.

The proof follows immediately from Theorem (3.2.2).

Theorem 4.1.6. *Let* $\{y_n\}_{n=-k}^{\infty}$ *be a solution of Eq.* (4.1.11), *then the following are true:*

1. *If k is odd and* $q > 1$, *then Eq.* (4.1.11) *has no solution of prime period two.*

2. If k is even, then Eq. (4.1.11) *has no solution of prime period two.*

Proof. Let

$$\ldots, \phi, \psi, \phi, \psi, \ldots$$

be a period two solution of Eq. (4.1.11), where ϕ *and* ψ are real numbers.

1. If k is odd, then $y_{n-k} = y_{n+1}$ and ϕ and ψ satisfy the following systems:

$$\phi = \frac{p+\phi}{q+d\psi+\phi} \quad \textit{and} \quad \psi = \frac{p+\psi}{q+d\phi+\psi} \tag{4.1.12}$$

simplifying the relation in Eq. (4.1.12) we obtain,

$$(\phi - \psi)\left[q - 1 + \phi + \psi\right] = 0$$

as $\phi \neq \psi$ then $\phi + \psi = 1 - q$, when $q > 1$
this is obvious contradiction.

2. If k is even, then $y_{n-k} = y_n$ and ϕ, ψ satisfy the following systems:

$$\phi = \frac{p+\psi}{q+d\psi+\psi} \quad \textit{and} \quad \psi = \frac{p+\phi}{q+d\phi+\phi}. \tag{4.1.13}$$

Simplifying relation (4.1.13) we obtain ,

$$q(\phi - \psi) = (\psi - \phi)$$

$$\Longrightarrow (\phi - \psi)\left[q + 1\right] = 0$$

as $q \neq -1$ then $\phi = \psi$.

So, Eq. (4.1.11) has no prime period two solution.

□

Theorem 4.1.7. *Assume that* $p > q+d$ *and* $\bar{y} \leq 1$ *then the equilibrium point* $\bar{y}$ *of Eq.* (4.1.11) *is globally attractive on the interval* $\left[\frac{q+d}{p}, 1\right]$.

Proof. When $p > q + d$ and $x \leq 1$ then the function

$$f(x, y) = \frac{p + y}{q + dx + y}$$

is decreasing in both arguments.
And as the Eq. (4.1.11) has no solution of prime period two, then by using Theorem (3.4.2) we see that, the equilibrium point $\bar{y}$ is globally attractor. □

Theorem 4.1.8. *Assume that* $p > q + d$ *and* $\bar{y} \leq 1$ *then every oscillatory solution of Eq.* (4.1.11) *has semi cycles of length at most* k.

The proof follows from Theorem (3.3.6).

4.1.4 The case $\alpha = 0$: $x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{A + Bx_n + Cx_{n-k}}$

Lemma 4.1.4. *The change of variables* $x_n = \frac{\gamma}{C} y_n$ *reduces Eq.* (4.1.6) *to the difference equation*

$$y_{n+1} = \frac{py_n + y_{n-k}}{q + dy_n + y_{n-k}} \tag{4.1.14}$$

where $p = \frac{\beta}{\gamma}$, $q = \frac{A}{\gamma}$, *and* $d = \frac{B}{C}$ *and the initial conditions* $y_{-k}, \ldots, y_0$ *are arbitrary nonnegative real numbers.*

Proof. Substitute $x_n = \frac{\gamma}{C} y_n$ in Eq. (4.1.14), we get

$$\frac{\gamma}{C} y_{n+1} = \frac{\frac{\beta\gamma}{C} y_n + \gamma \frac{\gamma}{C} y_{n-k}}{A + \frac{\beta\gamma}{C} y_n + \frac{C\gamma}{C} y_{n-k}}$$

cancel $\frac{\gamma}{C}$ from both side

$$\Longrightarrow y_{n+1} = \frac{\gamma \left[\frac{\beta}{\gamma} y_n + y_{n-k}\right]}{\gamma \left[\frac{A}{\gamma} + \frac{\beta}{C} y_n + y_{n-k}\right]}$$

set $p = \frac{\beta}{\gamma}$, $q = \frac{A}{\gamma}$ and $d = \frac{B}{C}$ we obtain Eq. (4.1.14).

□

The Eq. (4.1.14)has two equilibrium points

$$\bar{y} = 0 \ \ and \ \ \bar{y} = \frac{p+1-q}{(d+1)}.$$

And the linearized equation is

$$z_{n+1} - \left(\frac{pq+(p-d)\bar{y}}{(q+d\bar{y}+\bar{y})^2}\right) z_n + \left(\frac{q+(d-p)\bar{y}}{(q+d\bar{y}+\bar{y})^2}\right) z_{n-k} = 0.$$

Theorem 4.1.9. *Assume that $p > q+d$ and $p > d$, $d > 1$ and $d\bar{y} > p$ then the equilibrium point*

$$\bar{y} = \frac{p+1-q}{(d+1)}$$

is locally stable.

The proof follows immediately from Theorem (3.2.2).

Theorem 4.1.10. *Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq.* (4.1.14), *then Eq.* (4.1.14) *has no solution of prime period two in the following two cases:*

1. If k is odd and $p+q > 1$.

2. If k is even and $q \neq 1+p$.

Proof. Let

$$\ldots, \phi, \psi, \phi, \psi, \ldots$$

be a period two solution of Eq. (4.1.14), where ϕ *and* ψ are real numbers.

1. If k is odd, then $y_{n-k} = y_{n+1}$ then ϕ and ψ satisfy the following systems:

$$\phi = \frac{p\psi+\phi}{q+d\psi+\phi} \ \ and \ \ \psi = \frac{p\phi+\psi}{q+d\phi+\psi} \tag{4.1.15}$$

Thus, we have

$$(\psi - \phi)\,[q - 1 + p + \phi + \psi] = 0$$

if $\phi \neq \psi$ then we have $\phi + \psi = 1 - (p+q)$, when $p + q > 1$ then this is a contradiction as ϕ *and* ψ must be positive.

2. If k is even, then $y_{n-k} = y_n$ and ϕ, ψ satisfy the following systems:

$$\phi = \frac{p\psi + \psi}{q + d\psi + \psi} \quad and \quad \psi = \frac{p\phi + \phi}{q + d\phi + \phi} \tag{4.1.16}$$

hence, we have

$$(\phi - \psi)\,[p + q + 1] = 0$$

when $q \neq -(1+p)$ then $\phi = \psi$.

So, Eq. (4.1.14) has no prime period two solution.

□

Theorem 4.1.11. *Assume that $p > q + d$ where $p > d$ then the equilibrium point $\bar{y}$ of Eq. (4.1.14) is globally attractive on the interval $\left[1, \frac{p}{q+d}\right]$.*

Proof. When $p > q + d$ and $p > d$ then the function

$$f(x, y) = \frac{px + y}{q + dx + y}$$

is decreasing in y where $y \in \left[1, \frac{p}{q+d}\right]$.
And the function $f(x, y)$ is increasing in x when $p > q + d$ *and* $p > d$.

Then by using Theorem (3.4.1) we can see that, the equilibrium point $\bar{y} = \frac{p+1-q}{(d+1)}$ is globally attractive on the interval $\left[1, \frac{p}{q+d}\right]$.

□

Theorem 4.1.12. *Assume that $p > q + d$ where $p > q$ then every oscillatory solution of Eq.* (4.1.14) *has semi-cycles of length at least k.*

The proof follows from Theorem (3.3.3).

4.2 Two Parameters are zero

In this section we will study the character of solution of Eq. (3.1.1) where two parameters are zero. There are fifteen cases for this equation, namely:

$$x_{n+1} = \frac{\gamma x_{n-k}}{A + Bx_n + Cx_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{4.2.1}$$

$$x_{n+1} = \frac{\beta x_n}{A + Bx_n + Cx_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{4.2.2}$$

$$x_{n+1} = \frac{\alpha}{A + Bx_n + Cx_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{4.2.3}$$

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{A}, \qquad n = 0, 1, 2, \ldots \tag{4.2.4}$$

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{Bx_n}, \qquad n = 0, 1, 2, \ldots \tag{4.2.5}$$

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{Cx_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{4.2.6}$$

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{4.2.7}$$

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{A + Cx_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{4.2.8}$$

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{A + Bx_n}, \qquad n = 0, 1, 2, \ldots \tag{4.2.9}$$

$$x_{n+1} = \frac{\alpha + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{4.2.10}$$

$$x_{n+1} = \frac{\alpha + \gamma x_{n-k}}{A + Cx_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{4.2.11}$$

$$x_{n+1} = \frac{\alpha + \gamma x_{n-k}}{A + Bx_n}, \qquad n = 0, 1, 2, \ldots \tag{4.2.12}$$

$$x_{n+1} = \frac{\alpha + \beta x_n}{Bx_n + Cx_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{4.2.13}$$

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + Cx_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{4.2.14}$$

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + Bx_n}, \qquad n = 0, 1, 2, \ldots \tag{4.2.15}$$

Where the parameters $\alpha,\ \beta,\ \gamma$ *and* $A,\ B,\ C$ are non-negative real numbers and the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are arbitrary real numbers, and the denominator is nonzero.

Of these equations, Eq. (4.2.4) is linear difference equation.
Eq. (4.2.15) is a Riccati equation.
The positive equilibrium point of Eq. (4.2.15) is globally asymptotically stable.

4.2.1 The Case α=β = 0 : $x_{n+1} = \frac{\gamma x_{n-k}}{A+Bx_n+Cx_{n-k}}$

Lemma 4.2.1. *The change of variables $x_n = \frac{\gamma}{C} y_n$ reduces Eq.* (4.2.1) *to the difference equation*

$$y_{n+1} = \frac{y_{n-k}}{p + qy_n + y_{n-k}} \tag{4.2.16}$$

where $p = \frac{A}{\gamma}$ $q = \frac{B}{C}$ and the initial conditions $y_{-k}, \ldots, y_0$ are arbitrary non-negative real numbers.

Proof. Substitute $x_n = \frac{\gamma}{C} y_n$ in Eq. (4.2.1), to get

$$\frac{\gamma}{C} y_{n+1} = \frac{\frac{\gamma\gamma}{C} y_{n-k}}{A + \frac{\gamma B}{C} y_n + \frac{C\gamma}{C} y_{n-k}}$$

then

$$y_{n+1} = \frac{\gamma y_{n-k}}{\gamma\left[\frac{A}{\gamma} + \frac{B}{C}y_n + y_{n-k}\right]}$$

set $p = \frac{A}{\gamma}$, $q = \frac{B}{C}$ to get Eq. (4.2.16). □

The equilibrium points of Eq. (4.2.16) are $\bar{y} = 0$ *and* $\bar{y} = \frac{1-p}{1+q}$.

And the linearized equation

$$z_{n+1} + \frac{q\bar{y}}{(p+q\bar{y}+\bar{y})^2} z_n + - \left(\frac{p+q\bar{y}}{(p+q\bar{y}+\bar{y})^2}\right) z_{n-k} = 0.$$

Theorem 4.2.1. *The equilibrium point $\bar{y} = 0$ is locally stable when $p > 1$. When $p < 1$ then the equilibrium point $\bar{y} = \frac{1-p}{1+q}$ is unstable.*

The proof follows immediately from Theorem (3.2.2).

Theorem 4.2.2. *Let $\{y_n\}_{n=-k}^{\infty}$ be a nonnegative solution of Eq.* (4.2.16)*, then the following are true:*

- *If k is even, then Eq.* (4.2.16) *has no solution of prime period two.*

- *If k is odd, then Eq.* (4.2.16) *has no solution of prime period two when $p > 1$.*

Proof. Let

$$\ldots, \phi, \psi, \phi, \psi, \ldots$$

be a period two solution of Eq. (4.2.16), where ϕ *and* ψ are positive and distinct, then

- If k is even, then we have:

$$\phi = \frac{\psi}{p + q\psi + \psi} \quad \textit{and} \quad \psi = \frac{\phi}{p + q\phi + \phi}$$

so, we obtain

$$p(\phi - \psi) = \psi - \phi \Longrightarrow (\phi - \psi)(p+1) = 0$$

as $p \neq -1$, then $\phi = \psi$, and this is a contradiction.

- If k is odd, then we have

$$\phi = \frac{\phi}{p + q\psi + \phi} \quad and \quad \psi = \frac{\psi}{p + q\phi + \psi}$$

hence,

$$(\phi - \psi)\left[p + (\phi + \psi) - 1\right] = 0$$

$\Longrightarrow \phi + \psi = 1 - p$ when $p > 1$ then $\phi = \psi$ and this is a contradiction, as $\phi,\ \psi$ are positive hence, $\phi = \psi$.

So, Eq. (4.2.16) has no prime period two solution when k is odd and $p > 1$ or k is even.

□

Theorem 4.2.3. *Assume that $p > 1$ then the equilibrium point $\bar{y} = 0$ is globally attractive.*

Proof. Let

$$f(x, y) = \frac{y}{p + qx + y}$$

where $f : (0, \infty) \times (0, \infty) \longrightarrow (0, \infty)$ is continuous function.
As $f(x, y)$ is decreasing in x and increasing in y, $\forall\ x, y \in (0, \infty)$, then by using Theorem (3.4.3), we can prove that the equilibrium point $\bar{y}$ is globally attractive. □

Theorem 4.2.4. *Every solution of Eq.* (4.2.16) *has semi cycles of length k.*

Proof. As $f(x, y)$ is decreasing in x and increasing in y, so by using Theorem (3.3.4), we can prove that every solution of Eq. 4.2.16 has semi-cycles of length k.

□

4.2.2 The Case $\alpha=\gamma = 0$: $x_{n+1} = \frac{\beta x_n}{A+Bx_n+Cx_{n-k}}$

Lemma 4.2.2. *The change of variables $x_n = \frac{\beta}{B}y_n$ reduces Eq. (4.2.2) to the difference equation*

$$y_{n+1} = \frac{y_n}{p + qy_{n-k} + y_n} \tag{4.2.17}$$

where $p = \frac{A}{\beta}$, $q = \frac{C}{B}$ and the initial conditions $y_{-k}, \ldots, y_0$ are arbitrary nonnegative real numbers.

Proof. By substituting $x_n = \frac{\beta}{B}y_n$ in Eq. (4.2.2) we can get easily Eq. (4.2.17).

□

The equilibrium points of Eq. (4.2.17) are $\bar{y} = 0$ *and* $\bar{y} = \frac{1-p}{1+q}$.

And the linearized equation

$$z_{n+1} - \left(\frac{p + q\bar{y}}{(p + q\bar{y} + \bar{y})^2}\right) z_n + \frac{q\bar{y}}{(p + q\bar{y} + \bar{y})^2} z_{n-k} = 0.$$

Theorem 4.2.5. *The equilibrium point $\bar{y} = 0$ is locally stable when $p > 1$. And the equilibrium point $\bar{y} = \frac{1-p}{1+q}$ is unstable when $p < 1$.*

The proof follows immediately from Theorem (3.2.2).

Theorem 4.2.6. *The Eq. (4.2.17) has no solution of prime period two.*

Proof. Let

$$\ldots, \phi, \psi, \phi, \psi, \ldots$$

be a period two solution of Eq. (4.2.17), where ϕ *and* ψ are positive and distinct, then

- If k is even, then we have:

$$\phi = \frac{\psi}{p + q\psi + \psi} \quad \text{and} \quad \psi = \frac{\phi}{p + q\phi + \phi}. \tag{4.2.18}$$

Simplifying the relation in (4.2.18), we get

$$p(\phi - \psi) = \psi - \phi \Longrightarrow (\phi - \psi)(p + 1) = 0$$

since $\phi \neq \psi$ so we have $p = -1$, and this is a contradiction.

- If k is odd, then we have

$$\phi = \frac{\psi}{p + q\phi + \psi} \quad \textit{and} \quad \psi = \frac{\phi}{p + q\psi + \phi}. \tag{4.2.19}$$

Simplify the relation in (4.2.19), and we obtain

$$(\phi - \psi)\left[p + 1 + q(\phi + \psi)\right] = 0$$

since $\phi \neq \psi$ then we have $\phi + \psi = \frac{-(p+1)}{q}$, and this is a contradiction as ϕ, ψ are positive hence $\phi = \psi$

So, Eq. (4.2.17) has no solution of prime period two.

□

Theorem 4.2.7. *Assume that $p > 1$ then the equilibrium point $\bar{y} = 0$ of Eq.* (4.2.17) *is globally attractive.*

Proof. Let

$$f(x, y) = \frac{x}{p + qy + x}$$

where $f : (0, \infty) \times (0, \infty) \longrightarrow (0, \infty)$ is continuous function.
As $f(x, y)$ is increasing in x and decreasing in y, $\forall\ x, y \in (0, \infty)$, then by using Theorem (3.4.1), we can prove that the equilibrium point $\bar{y}$ is globally attractive. □

Theorem 4.2.8. *Every solution of Eq.* (4.2.17) *has semi-cycles of length at least k.*

Proof. As $f(x, y)$ is increasing in x and decreasing in y, then the proof follows immediately by using Theorem (3.3.3).

□

4.2.3 The Case $\beta = \gamma = 0$: $x_{n+1} = \frac{\alpha}{A + Bx_n + Cx_{n-k}}$

Lemma 4.2.3. *The change of variables $x_n = \frac{\sqrt{\alpha}}{y_n}$ reduces Eq. (4.2.3) to the difference equation*

$$y_{n+1} = p + \frac{\beta}{y_n} + \frac{C}{y_{n-k}} \tag{4.2.20}$$

where $p = \frac{A}{\sqrt{\alpha}}$ and the initial conditions $y_{-k}, \ldots, y_0$ are arbitrary nonnegative real numbers.

Proof. Substitute $x_n = \frac{\sqrt{\alpha}}{y_n}$ in Eq. (4.2.3) we obtain,

$$\frac{\sqrt{\alpha}}{y_{n+1}} = \frac{\alpha}{A + \frac{B\sqrt{\alpha}}{y_n} + \frac{C\sqrt{\alpha}}{y_{n-k}}}$$

hence,

$$\frac{1}{y_{n+1}} = \frac{1}{\frac{A}{\alpha} + \frac{B}{y_n} + \frac{C}{y_{n-k}}}$$

so,

$$y_{n+1} = \frac{A}{\sqrt{\alpha}} + \frac{B}{y_n} + \frac{C}{y_{n-k}}$$

set, $\frac{A}{\sqrt{\alpha}} = p$ we get Eq. (4.2.20).

□

The only positive equilibrium point of Eq. (4.2.20) is
$\bar{y} = p + \sqrt{p^2 + 4(B + C)}$.

And the linearized equation

$$z_{n+1} + \frac{B}{\bar{y}^2} z_n + \frac{C}{\bar{y}^2} z_{n-k} = 0.$$

Theorem 4.2.9. *The equilibrium point $\bar{y} = p + \sqrt{p^2 + 4(B + C)}$ of Eq. (4.2.20) is locally stable.*

The proof follows immediately from Theorem (3.2.2).

Theorem 4.2.10. *Let* $\{y_n\}_{n=-k}^{\infty}$ *be a nonnegative solution of Eq.* (4.2.20), *then the following are true:*

- *If* k *is even, then Eq.* (4.2.20) *has no solution of prime period two.*

- *If* k *is odd, then Eq.* (4.2.20) *has prime period two solution and this solution has the form*

$$\ldots, \phi, \frac{B-C}{\phi}, \phi, \frac{B-C}{\phi}, \ldots$$

Proof. Let

$$\ldots, \phi, \psi, \phi, \psi, \ldots$$

be a period two solution of Eq. (4.2.20), where ϕ *and* ψ are positive and distinct, then

- If k is even, then we have:

$$\phi = p + \frac{B}{\psi} + \frac{C}{\psi} \quad \textit{and} \quad \psi = p + \frac{B}{\phi} + \frac{C}{\phi} \tag{4.2.21}$$

  Simplifying the relation in (4.2.21), we obtain

$$p(\psi - \phi) = 0 \Longrightarrow \phi = \psi$$

  so, when k is even then the Eq. (4.2.20) has no solution of prime period two.

- If k is odd, then we get

$$\phi = p + \frac{B}{\psi} + \frac{C}{\phi} \quad \textit{and} \quad \psi = p + \frac{B}{\phi} + \frac{C}{\psi} \tag{4.2.22}$$

  from relation (4.2.22) we get,

$$(\phi - \psi)\left[\phi\psi + C - B\right] = 0$$

  since $\phi \neq \psi$ then $\psi = \frac{B-C}{\phi}$.

So, when k is odd, then the prime period two solution of Eq. (4.2.20) have the form.

$$\ldots, \phi, \frac{B-C}{\phi}, \phi, \frac{B-C}{\phi}, \ldots$$

and this complete the proof.

□

Theorem 4.2.11. *The equilibrium point $\bar{y} = p + \sqrt{p^2 + 4(B+C)}$ of Eq.* (4.2.20) *is globally attractive.*

Proof. The function

$$f(x, y) = p + \frac{B}{x} + \frac{C}{y}$$

is decreasing in both arguments, and the Eq. (4.2.20) has no solution of prime period two when k is even, then by Theorem (3.4.2) the equilibrium point $\bar{y}$ is globally attractive. □

Theorem 4.2.12. *Every solution of Eq.* (4.2.20) *has semi-cycles of length at most k.*

Proof. As $f(x, y)$ is decreasing in both arguments, then the proof follows from Theorem (3.3.6).

□

4.2.4 The Case A=C = 0 : $x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-k}}{Bx_n}$

Lemma 4.2.4. *The Eq.* (4.2.5) *is reduced by the change of variables $x_n = \frac{\beta}{B} + \frac{\beta}{B} y_n$ to the difference equation*

$$y_{n+1} = \frac{p + q y_{n-k}}{1 + y_n} \tag{4.2.23}$$

where $p = \frac{\alpha B}{\beta^2} + \frac{\gamma}{\beta}$ and $q = \frac{\gamma}{\beta}$, and the initial conditions $y_{-k}, \ldots, y_0$ are nonnegative real numbers.

Proof. Set $x_n = \frac{\beta}{B} + \frac{\beta}{B} y_n$,then substitute the value of x_n in Eq. (4.2.5), we get

$$\frac{\beta}{B} y_{n+1} + \frac{\beta}{B} = \frac{\alpha + \beta \frac{\beta}{B}\left[1 + y_n\right] + \frac{\gamma\beta}{B}\left[1 + y_{n-k}\right]}{\frac{B\beta}{B}\left[1 + y_n\right]}$$

thus,

$$\frac{\beta}{B}\left[1 + y_{n+1}\right] = \frac{\frac{\beta^2}{B}\left[\frac{\alpha B}{\beta^2} + (1 + y_n) + \frac{\gamma}{\beta}(1 + y_{n-k})\right]}{\beta\left[1 + y_n\right]}$$

$$\Longrightarrow y_{n+1} = \frac{(\frac{\alpha B}{\beta^2} + \frac{\gamma}{\beta}) + \frac{\gamma}{\beta} y_{n-k}}{1 + y_n}.$$

By letting, $p = \frac{\alpha B}{\beta^2} + \frac{\gamma}{\beta}$ and $q = \frac{\gamma}{\beta}$, we get Eq. (4.2.23). □

The Eq. (4.2.23) was investigated in [9], by Douraki, Dehghan and Razzaghi.

4.2.5 The Case B=A = 0 : $x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{C x_{n-k}}$

Lemma 4.2.5. *The change of variables $x_n = \frac{\gamma}{C} + \frac{\gamma}{C} y_n$ reduces Eq. (4.2.6) into equation*

$$y_{n+1} = \frac{p + q y_n}{1 + y_{n-k}} \tag{4.2.24}$$

where $p = \frac{\alpha C + \beta\gamma}{\gamma^2}$ and $q = \frac{\beta}{\gamma}$ with $p, q \in (0, \infty)$ and the initial conditions $y_{-k}, \ldots, y_0$ are nonnegative real numbers.

Proof. Substitute $x_n = \frac{\gamma}{C} + \frac{\gamma}{C} y_n$ in Eq. (4.2.6) we get

$$\frac{\gamma}{C} + \frac{\gamma}{C} y_{n+1} = \frac{\alpha + \beta\left[\frac{\gamma}{C} + \frac{\gamma}{C} y_n\right] + \gamma\left[\frac{\gamma}{C} + \frac{\gamma}{C} y_{n-k}\right]}{C\frac{\gamma}{C}\left[y_{n-k} + 1\right]}$$

then

$$y_{n+1} = \frac{\frac{\alpha C}{\gamma^2} + \frac{\beta}{\gamma}(1 + y_n) + (1 + y_{n-k})}{(y_{n-k} + 1)} - 1$$

$$\Longrightarrow y_{n+1} = \frac{\frac{\alpha C}{\gamma^2} + \frac{\beta}{\gamma}(1 + y_n)}{y_{n-k} + 1}$$

set $p = \frac{\alpha C+\beta\gamma}{\gamma^2}$ and $q = \frac{\beta}{\gamma}$, we get Eq. (4.2.24).

The Eq. (4.2.24) was investigated in [7], by Dehghan and Sebdani.

□

Eq. (4.2.7) was investigated by S.Abu Baha in [1], and Eq. (4.2.8) was investigated by A.Farhat in [13].

4.2.6 The Case α=C = 0 : $x_{n+1} = \frac{\beta x_n+\gamma x_{n-k}}{A+Bx_n}$

Lemma 4.2.6. *The change of variables* $x_n = \frac{\beta}{B}y_n$, *reduces Eq.* (4.2.9) *to the difference equation*

$$y_{n+1} = \frac{y_n + py_{n-k}}{q + y_n} \tag{4.2.25}$$

where $p = \frac{\gamma}{\beta}$ *and* $q = \frac{A}{\beta}$ *with* $p, q \in (0, \infty)$ *and the initial conditions* $y_{-k}, \ldots, y_0$ *are nonnegative real numbers.*

Proof. Substitute $x_n = \frac{\beta}{B}y_n$ in Eq. (4.2.9),
we get

$$\frac{\beta}{B}y_{n+1} = \frac{\beta\frac{\beta}{B}y_n + \gamma\frac{\beta}{B}y_{n-k}}{A + B\frac{\beta}{B}y_n}$$

then

$$y_{n+1} = \frac{\beta\left[y_n + \frac{\gamma}{\beta}y_{n-k}\right]}{\beta\left[\frac{A}{\beta} + y_n\right]}.$$

By, letting $p = \frac{\gamma}{\beta}$ *and* $q = \frac{A}{\beta}$ we get Eq. (4.2.25).

□

Eq. (4.2.25) was investigated in [23] by Dehghan and Sebdani.

4.2.7 The Case β =A = 0 : $x_{n+1} = \frac{\alpha+\gamma x_{n-k}}{Bx_n+Cx_{n-k}}$

Lemma 4.2.7. *The change of variables $x_n = \frac{\gamma}{C}y_n$ reduces Eq. (4.2.10) to the difference equation*

$$y_{n+1} = \frac{p + y_{n-k}}{qy_n + y_{n-k}} \tag{4.2.26}$$

where $p = \frac{\alpha C}{\gamma^2}$ and $q = \frac{B}{C}$ and the initial conditions $y_{-k}, \ldots, y_0$ are nonnegative real numbers.

Proof. Substitute $x_n = \frac{\gamma}{C}y_n$ in Eq. (4.2.10)
we get,

$$\frac{\gamma}{C}y_{n+1} = \frac{\alpha + \gamma\frac{\gamma}{C}y_{n-k}}{B\frac{\gamma}{C}y_n + C\frac{\gamma}{C}y_{n-k}}$$

then

$$y_{n+1} = \frac{\frac{\alpha C}{\gamma^2} + y_{n-k}}{\frac{B}{C}y_n + y_{n-k}}$$

set $p = \frac{\alpha C}{\gamma^2}$ and $q = \frac{B}{C}$, we get Eq. (4.2.26).

□

The Eq. (4.2.26) was investigated by Devalut, Ladas and Kosmala in [8].

4.2.8 The Case β =B = 0 : $x_{n+1} = \frac{\alpha+\gamma x_{n-k}}{A+Cx_{n-k}}$

Lemma 4.2.8. *The change of variables $x_n = \frac{\gamma}{C}y_n$ reduces Eq. (4.2.11) to the difference equation*

$$y_{n+1} = \frac{p + y_{n-k}}{q + y_{n-k}} \tag{4.2.27}$$

where $p = \frac{\alpha C}{\gamma^2}$ and $q = \frac{A}{\gamma}$ and the initial conditions $y_{-k}, \ldots, y_0$ are arbitrary nonnegative real numbers.

Proof. Substitute $x_n = \frac{\gamma}{C}y_n$ in Eq. (4.2.11), to get

$$\frac{\gamma}{C}y_{n+1} = \frac{\alpha + \frac{\gamma\gamma}{C}y_{n-k}}{A + \frac{C\gamma}{C}y_{n-k}}$$

then

$$y_{n+1} = \frac{\gamma\left[\frac{\alpha C}{\gamma^2} + y_{n-k}\right]}{\gamma\left[\frac{A}{\gamma} + y_{n-k}\right]}$$

set $p = \frac{\alpha C}{\gamma^2}$ and $q = \frac{A}{\gamma}$ to get Eq. (4.2.27). □

The only positive equilibrium point of Eq. (4.2.27) is $\bar{y} = \frac{(1-q)+\sqrt{(q-1)^2+4p}}{2}$.

And the linearized equation

$$z_{n+1} - \frac{(p-q)}{(q-\bar{y})^2} z_{n-k} = 0.$$

Theorem 4.2.13. *The equilibrium point* $\bar{y} = \frac{(1-q)+\sqrt{(q-1)^2+4p}}{2}$ *is locally stable when* $p > q$.

The proof follows immediately from Theorem (3.2.2).

Theorem 4.2.14. *Let* $\{y_n\}_{n=-k}^{\infty}$ *be a nonnegative solution of Eq.* (4.2.27)*, then the following are true:*

- *If* k *is even, then Eq.* (4.2.27) *has no solution of prime period two.*

- *If* k *is odd and* $q < 1$*, then Eq.* (4.2.27) *has a solution of prime period two of the form,*

$$\ldots, \phi, 1-q-\phi, \phi, 1-q-\phi, \phi, \ldots$$

Proof. Let

$$\ldots, \phi, \psi, \phi, \psi, \ldots$$

be a period two solution of Eq. (4.2.27), where ϕ *and* ψ are positive and distinct, then

- If k is even, then we have the following systems:

$$\phi = \frac{p+\psi}{q+\psi} \quad and \quad \psi = \frac{p+\phi}{q+\phi} \tag{4.2.28}$$

Simplifying Eq. (4.2.28) we obtain,

$$(\phi - \psi)\,[q+1] = 0$$

as $q \neq -1$, so $\phi = \psi$.

- If k is odd, then we have

$$\phi = \frac{p+\phi}{q+\phi} \quad and \quad \psi = \frac{p+\psi}{p+\psi} \tag{4.2.29}$$

simplifying the relation in Eq. (4.2.29) we get,

$$(\phi - \psi)\,[q + (\phi + \psi) - 1] = 0$$

$\Longrightarrow \phi + \psi = 1 - q$ when $q > 1$ then $\phi = \psi$, and when $q < 1$ then Eq. (4.2.27) has prime period two solution of the form

$$\ldots, \phi, 1-q-\phi, \phi, 1-q-\phi, \phi, \ldots$$

□

4.2.9 The Case β =C = 0 : $x_{n+1} = \frac{\alpha+\gamma x_{n-k}}{A+Bx_n}$

Lemma 4.2.9. *The change of variables $x_n = \frac{A}{B} y_n$ reduces Eq.* (4.2.12) *to the difference equation*

$$y_{n+1} = \frac{p + q y_{n-k}}{1 + y_n} \tag{4.2.30}$$

where $p = \frac{\alpha\beta}{A^2}$ and $q = \frac{\gamma}{A}$ and the initial conditions $y_{-k}, \ldots, y_0$ are nonnegative real numbers.

Proof. Substitute $x_n = \frac{A}{B} y_n$ in Eq. (4.2.12)
we get,

$$\frac{A}{B} y_{n+1} = \frac{\alpha + \gamma \frac{A}{B} y_{n-k}}{A + B \frac{A}{B} y_n}$$

then

$$y_{n+1} = \frac{\frac{\alpha B}{A^2} + \frac{\gamma}{A} y_{n-k}}{1 + y_n}$$

set $p = \frac{\alpha B}{A^2}$ and $q = \frac{\gamma}{A}$, we get Eq. (4.2.30).

□

The Eq. (4.2.30) was investigated by M.J.Douraki, M.Dehghan and M.Razzaghi in [9].

4.2.10 The Case $\gamma = A = 0$: $x_{n+1} = \frac{\alpha + \beta x_n}{B x_n + C x_{n-k}}$

Lemma 4.2.10. *The change of variables* $x_n = \frac{\beta}{B} y_n$ *reduces Eq.* (4.2.13) *to the difference equation*

$$y_{n+1} = \frac{p + y_n}{y_n + q y_{n-k}} \tag{4.2.31}$$

where $p = \frac{\alpha\beta}{B^2}$ *and* $q = \frac{C}{B}$ *and the initial conditions* $y_{-k}, \ldots, y_0$ *are nonnegative real numbers.*

Proof. Substitute $x_n = \frac{\beta}{B} y_n$ in Eq. (4.2.13)
we get,

$$\frac{\beta}{B} y_{n+1} = \frac{\alpha + \beta \frac{\beta}{B} y_n}{B \frac{\beta}{B} y_n + C \frac{\beta}{B} y_{n-k}}$$

then

$$y_{n+1} = \frac{\frac{\alpha B}{\beta^2} + y_n}{y_n + \frac{C}{B} y_{n-k}}$$

set $p = \frac{\alpha B}{\beta^2}$ and $q = \frac{C}{B}$, we get Eq. (4.2.31).

□

The Eq. (4.2.31) was investigated by M.J.Douraki and M.Dehghan in [5].

4.2.11 The Case γ= B = 0 : $x_{n+1} = \frac{\alpha+\beta x_n}{A+Cx_{n-k}}$

Lemma 4.2.11. *The change of variables $x_n = \frac{A}{C}y_n$ reduces Eq.* (4.2.14) *to the difference equation*

$$y_{n+1} = \frac{p + qy_n}{1 + y_{n-k}} \tag{4.2.32}$$

where $p = \frac{\alpha C}{A^2}$ and $q = \frac{B}{A}$ and the initial conditions $y_{-k}, \ldots, y_0$ are nonnegative real numbers.

Proof. Substitute $x_n = \frac{A}{C}y_n$ in Eq. (4.2.14)
we get,

$$\frac{A}{C}y_{n+1} = \frac{\alpha + \beta\frac{A}{C}y_n}{A + C\frac{A}{C}y_{n-k}}$$

then

$$y_{n+1} = \frac{\frac{\alpha C}{A^2} + \frac{\beta}{A}y_n}{1 + y_{n-k}}$$

set $p = \frac{\alpha C}{A^2}$ and $q = \frac{\beta}{A}$ we get Eq. (4.2.32).

□

The Eq. (4.2.32) was studied by M.Dehghan and M.Razzaghi in [7].

4.3 Three Parameters are zero

In this section we will study the character of solution of Eq. (3.1.1) where three parameters are zero. There are eighteen cases for this equation, namely:

$$x_{n+1} = \frac{\alpha + \beta x_n}{A}, \qquad n = 0, 1, 2, \ldots \tag{4.3.1}$$

$$x_{n+1} = \frac{\alpha + \beta x_n}{B x_n}, \qquad n = 0, 1, 2, \ldots \tag{4.3.2}$$

$$x_{n+1} = \frac{\alpha + \beta x_n}{C x_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{4.3.3}$$

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{A}, \qquad n = 0, 1, 2, \ldots \tag{4.3.4}$$

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{B x_n}, \qquad n = 0, 1, 2, \ldots \tag{4.3.5}$$

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{C x_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{4.3.6}$$

$$x_{n+1} = \frac{\alpha + \gamma x_{n-k}}{A}, \qquad n = 0, 1, 2, \ldots \tag{4.3.7}$$

$$x_{n+1} = \frac{\alpha + \gamma x_{n-k}}{B x_n}, \qquad n = 0, 1, 2, \ldots \tag{4.3.8}$$

$$x_{n+1} = \frac{\alpha + \gamma x_{n-k}}{C x_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{4.3.9}$$

$$x_{n+1} = \frac{\alpha}{B x_n + C x_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{4.3.10}$$

$$x_{n+1} = \frac{\alpha}{A + B x_n}, \qquad n = 0, 1, 2, \ldots \tag{4.3.11}$$

$$x_{n+1} = \frac{\alpha}{A + C x_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{4.3.12}$$

$$x_{n+1} = \frac{\beta x_n}{A + B x_n}, \qquad n = 0, 1, 2, \ldots \tag{4.3.13}$$

$$x_{n+1} = \frac{\beta x_n}{A + C x_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{4.3.14}$$

$$x_{n+1} = \frac{\beta x_n}{B x_n + C x_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{4.3.15}$$

$$x_{n+1} = \frac{\gamma x_{n-k}}{Bx_n + Cx_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{4.3.16}$$

$$x_{n+1} = \frac{\gamma x_{n-k}}{A + Bx_n}, \qquad n = 0, 1, 2, \ldots \tag{4.3.17}$$

$$x_{n+1} = \frac{\gamma x_{n-k}}{A + Cx_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{4.3.18}$$

Where the parameters $\alpha,\ \beta,\ \gamma$ *and* $A,\ B,\ C$ are non-negative real numbers and the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are arbitrary real numbers, and the denominator is nonzero.

Of these equations, Eqs. (4.3.1), (4.3.4) and (4.3.7) are linear difference equation. Eq. (4.3.2) is a Riccati equation.

4.3.1 The Case γ= A= B = 0 : $x_{n+1} = \frac{\alpha+\beta x_n}{Cx_{n-k}}$

Lemma 4.3.1. *The change of variables* $x_n = \frac{\beta}{C} y_n$ *reduces Eq.* (4.3.3) *to the difference equation*

$$y_{n+1} = \frac{p + y_n}{y_{n-k}} \tag{4.3.19}$$

where $p = \frac{\alpha C}{\beta^2}$*, and the initial conditions* $y_{-k}, \ldots, y_0$ *are arbitrary nonnegative real numbers.*

Proof. Substitute $x_n = \frac{\beta}{C} y_n$ in Eq. (4.3.3), we get

$$\frac{\beta}{C} y_{n+1} = \frac{\alpha + \frac{\beta\beta}{C} y_n}{C \frac{\beta}{C} y_{n-k}}.$$

Then

$$y_{n+1} = \frac{\beta \left[\frac{\alpha C}{\beta^2} + y_n \right]}{\beta y_{n-k}}$$

set, $p = \frac{\alpha C}{\beta^2}$ we get Eq. (4.3.19)

□

The Eq. (4.3.19) was studied by Alaweneh in [3].

4.3.2 The Case α =A =C = 0 : $x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{B x_n}$

Lemma 4.3.2. *The change of variables* $x_n = \frac{\gamma}{B} y_n$ *reduces Eq.* (4.3.5) *to the difference equation*

$$y_{n+1} = p + \frac{y_{n-k}}{y_n} \tag{4.3.20}$$

where $p = \frac{\beta}{\gamma}$*, and the initial conditions* $y_{-k}, \ldots, y_0$ *are arbitrary nonnegative real numbers.*

Proof. Substitute $x_n = \frac{\gamma}{B} y_n$ in Eq. (4.3.5), to get

$$\frac{\gamma}{B} y_{n+1} = \frac{\beta \frac{\gamma}{\beta} y_n + \gamma \frac{\gamma}{B} y_{n-k}}{B \frac{\gamma}{B} y_n}$$

then

$$y_{n+1} = \frac{\frac{\beta}{\gamma} y_n + y_{n-k}}{y_n} \Longrightarrow \frac{\beta}{\gamma} + \frac{y_{n-k}}{y_n}$$

set $p = \frac{\beta}{\gamma}$ to get Eq. (4.3.20). □

The Eq. (4.3.20) was investigated in [19] by M.Saleh and M.Aloqeili, and in [15] by W.S.He and X.X.Yan.

4.3.3 The Case α =A= B = 0 : $x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{C x_{n-k}}$

Lemma 4.3.3. *The change of variables* $x_n = \frac{\beta}{C} y_n$ *reduces Eq.* (4.3.6) *to the difference equation*

$$y_{n+1} = p + \frac{y_n}{y_{n-k}} \tag{4.3.21}$$

where $p = \frac{\gamma}{B}$*, and the initial conditions* $y_{-k}, \ldots, y_0$ *are nonnegative real numbers.*

Proof. Substitute $x_n = \frac{\beta}{C} y_n$ in Eq. (4.3.6) to get:

$$\frac{\beta}{C} y_{n+1} = \frac{\beta \frac{\beta}{C} y_n + \gamma \frac{\beta}{C} y_{n-k}}{C \frac{\beta}{C} y_{n-k}}$$

then

$$y_{n+1} = \frac{y_n + \frac{\gamma}{\beta} y_{n-k}}{y_{n-k}} \Longrightarrow y_{n+1} = \frac{y_n}{y_{n-k}} + \frac{\gamma}{\beta}$$

set $p = \frac{\gamma}{\beta}$, we get Eq. (4.3.21).

□

The Eq. (4.3.21) was studied by M.Saleh and M.Aloqeili in [20].

4.3.4 The Case β = A = C = 0 : $x_{n+1} = \frac{\alpha+\gamma x_{n-k}}{Bx_n}$

Lemma 4.3.4. *The change of variables* $x_n = \frac{\gamma}{B} y_n$ *reduces Eq.* (4.3.8) *to the difference equation*

$$y_{n+1} = \frac{p + y_{n-k}}{y_n} \tag{4.3.22}$$

where $p = \frac{\alpha\beta}{\gamma^2} \in (0, \infty)$, *and the initial conditions* $y_{-k}, \ldots, y_0$ *are arbitrary nonnegative real numbers.*

Proof. Substituting $x_n = \frac{\gamma}{B} y_n$ into Eq. (4.3.8), we can easily get Eq. (4.3.22)

□

Alaweneh in [3] investigated this equation.

4.3.5 The Case β =A =B = 0 : $x_{n+1} = \frac{\alpha+\gamma x_{n-k}}{Cx_{n-k}}$

Lemma 4.3.5. *The change of variables* $x_n = \frac{\gamma}{C} y_n$ *reduces Eq.* (4.3.9) *to the difference equation*

$$y_{n+1} = \frac{p + y_{n-k}}{y_{n-k}} \tag{4.3.23}$$

where $p = \frac{\alpha C}{\gamma^2}$, *and the initial conditions* $y_{-k}, \ldots, y_0$ *are arbitrary nonnegative real numbers.*

Proof. Substitute $x_n = \frac{\gamma}{C} y_n$ in Eq. (4.3.9), to get

$$\frac{\gamma}{C} y_{n+1} = \frac{\alpha + \frac{\gamma\gamma}{C} y_{n-k}}{\frac{C\gamma}{C} y_{n-k}}$$

then

$$y_{n+1} = \frac{\gamma\left[\frac{\alpha C}{\gamma^2} + y_{n-k}\right]}{\gamma y_{n-k}}$$

set $p = \frac{\alpha C}{\gamma^2}$ to get Eq. (4.3.23).

□

The only positive equilibrium point of Eq. (4.3.23) is
$\bar{y} = 1 + \sqrt{1+4p}$.

And the linearized equation

$$z_{n+1} + \frac{p}{\bar{y}^2} z_{n-k} = 0.$$

Theorem 4.3.1. *The equilibrium point $\bar{y} = 1 + \sqrt{1+4p}$ is locally stable.*

The proof follows from Theorem (3.2.2).

4.3.6 The Case $\beta = \gamma$ =A = 0 : $x_{n+1} = \frac{\alpha}{Bx_n + Cx_{n-k}}$

Lemma 4.3.6. *The change of variables $x_n = \frac{\sqrt{\alpha}}{y_n}$ reduces Eq.* (4.3.10) *to the difference equation*

$$y_{n+1} = \frac{B}{y_n} + \frac{C}{y_{n-k}} \tag{4.3.24}$$

where the initial conditions $y_{-k}, \ldots, y_0$ are non negative real numbers.

Proof. By Substituting $x_n = \frac{\sqrt{\alpha}}{y_n}$ into Eq. (4.3.10) we can get Eq. (4.3.24).

□

A.E.Alweneh studied Eq. (4.3.24) in [3].

4.3.7 The Case $\beta = \gamma = \mathbf{C} = \mathbf{0}$: $x_{n+1} = \frac{\alpha}{A+Bx_n}$

By the change of variables $x_n = \frac{A}{B}\, y_n$ Eq. (4.3.11) reduces to Riccati equation

$$y_{n+1} = \frac{p}{1+y_n} \qquad n = 0, 1, \ldots \tag{4.3.25}$$

Where $p = \frac{\alpha\beta}{A^2},\ p \in (0, \infty)$

Theorem 4.3.2. *The positive equilibrium point*

$$\bar{y} = \frac{-1+\sqrt{1+4p}}{2}$$

of Eq. (4.3.25) *is globally asymptotically stable.*

4.3.8 The Case $\beta = \gamma = \mathbf{B} = \mathbf{0}$: $x_{n+1} = \frac{\alpha}{A+Cx_{n-k}}$

Lemma 4.3.7. *The change of variables* $x_n = \frac{\sqrt{\alpha}}{y_n}$, *reduces Eq.* (4.3.12) *to the difference equation*

$$y_{n+1} = p + \frac{C}{y_{n-k}} \qquad n = 0, 1, \ldots \tag{4.3.26}$$

where $p = \frac{A}{\sqrt{\alpha}}$, *and the initial conditions* $y_{-k}, \ldots, y_0$ *are arbitrary nonnegative real numbers.*

Proof. Substitute $x_n = \frac{\sqrt{\alpha}}{y_n}$, we get

$$\frac{\sqrt{\alpha}}{y_{n+1}} = \frac{\alpha}{A + C\frac{\sqrt{\alpha}}{y_{n-k}}}$$

$$\Longrightarrow \frac{1}{y_{n+1}} = \frac{1}{\frac{A}{\sqrt{\alpha}} + \frac{C}{y_{n-k}}}$$

hence,

$$y_{n+1} = \frac{A}{\sqrt{\alpha}} + \frac{C}{y_{n-k}}$$

set $p = \frac{A}{\sqrt{\alpha}}$, we get Eq. (4.3.26).

□

The only positive equilibrium point is

$$\bar{y} = \frac{p + \sqrt{p^2 + 4C}}{2}.$$

And the linearized equation of this equilibrium point is

$$z_{n+1} + \frac{C}{\bar{y}^2} z_{n-k} = 0.$$

Theorem 4.3.3. *The equilibrium point* $\bar{y} = \frac{p+\sqrt{p^2+4C}}{2}$ *is locally stable.*

The proof follows immediately from Theorem (3.2.2)

Theorem 4.3.4. *Let* $\{y_n\}_{n=-k}^{\infty}$ *be a non negative solution of Eq.* (4.3.26)*, then the following are true.*

- *If* k *is even, then Eq.* (4.3.26) *has no solution of prime period two.*

- *If* k *is odd, then Eq.* (4.3.26) *has prime period two solution, and this solution take the form*

$$\ldots, \phi, p - \phi, \phi, p - \phi, \ldots$$

Proof. Let

$$\ldots, \phi, \psi, \phi, \psi, \ldots$$

be a period two solution of the Eq. (4.3.26) where ϕ, ψ are positive and distinct, then

- If k is even, then we have

$$\phi = p + \frac{C}{\psi} \quad and \quad \psi = p + \frac{C}{\phi}$$

then $p\phi + C = p\psi + C \implies p\phi = p\psi \implies \phi = \psi$ which is a contradiction.

- If k is odd, then

$$\phi = p + \frac{C}{\phi} \quad and \quad \psi = p + \frac{C}{\psi}$$

then we get, $(\phi - \psi)[\phi + \psi - p] = 0$
when $\phi \neq \psi$, this implies that $\phi + \psi = p$ so the period two solution must be of the form

$$\ldots, \phi, p - \phi, \phi, p - \phi, \ldots$$

which is complete the proof.

□

4.3.9 The Case $\alpha = \gamma$ =C = 0 : $x_{n+1} = \frac{\beta x_n}{A + Bx_n}$

The change of variables $x_n = \frac{1}{y_n}$ reduces Riccati Eq. (4.3.13) to the linear equation

$$y_{n+1} = \frac{A}{\beta} y_n + \frac{B}{\beta}, \qquad n = 0, 1, \ldots. \tag{4.3.27}$$

and Eq. (4.3.27) is linear first order difference equation.

4.3.10 The Case $\alpha = \gamma$ = B = 0 : $x_{n+1} = \frac{\beta x_n}{A + Cx_{n-k}}$

Lemma 4.3.8. *The change of variables* $x_n = \frac{\beta}{C} y_n$ *reduces Eq.* (4.3.14) *to the difference equation*

$$y_{n+1} = \frac{y_n}{p + y_{n-k}} \tag{4.3.28}$$

where $p = \frac{A}{\beta}$, *and the initial conditions* $y_{-k}, \ldots, y_0$ *are arbitrary nonnegative real numbers.*

Proof. Substitute $x_n = \frac{\beta}{C} y_n$ in Eq. (4.3.14), to get

$$\frac{\beta}{C} y_{n+1} = \frac{\frac{\beta\beta}{C} y_n}{A + \frac{C\beta}{C} y_{n-k}}$$

then

$$y_{n+1} = \frac{\beta y_n}{\beta\left[\frac{A}{\beta} + y_{n-k}\right]}$$

set $p = \frac{A}{\beta}$ to get Eq. (4.3.28). □

The Eq. (4.3.28) has two equilibrium points $\bar{y} = 0$ *and* $\bar{y} = 1 - p$.

And the linearized equation

$$z_{n+1} - \frac{(p+\bar{y})}{(p+\bar{y})^2} z_n + \frac{\bar{y}}{(p+\bar{y})^2} z_{n-k} = 0.$$

Theorem 4.3.5. *When $p > 1$ then the equilibrium point $\bar{y} = 0$ is locally stable.*

The proof follows immediately from Theorem (3.2.2).

Theorem 4.3.6. *The Eq.* (4.3.28) *has no solution of prime period two.*

Proof. Let

$$\ldots, \phi, \psi, \phi, \psi, \ldots$$

be a period two solution of Eq. (4.3.28), where ϕ *and* ψ are two arbitrary positive and distinct real numbers.

- If k is odd, then $y_{n+1} = y_{n-k}$ and $\phi,\ \psi$ satisfy the following systems:

$$\phi = \frac{\psi}{p+\phi} \quad and \quad \psi = \frac{\phi}{p+\psi} \tag{4.3.29}$$

  simplifying the relation in Eq. (4.3.29) to get,

$$(\psi - \phi)\left[p + (\phi + \psi) + 1\right] = 0$$

  $\Longrightarrow \phi + \psi = -(1+p)$ and this impossible.

  So $\phi = \psi$.

- If k is even, then $y_n = y_{n-k}$ and ϕ, ψ satisfy the following systems:

$$\phi = \frac{\psi}{p+\psi} \quad \textit{and} \quad \psi = \frac{\phi}{p+\phi} \tag{4.3.30}$$

Simplifying Eq. (4.3.30), we obtain

$$(\phi - \psi)\,[p+1] = 0$$

$p \neq -1$, so $\phi = \psi$.

The proof is complete.

□

Theorem 4.3.7. *When $p > 1$ then the equilibrium point $\bar{y} = 0$ of Eq.* (4.3.28) *is globally attractive.*

Proof. As the function

$$f(x,y) = \frac{x}{p+y}$$

is increasing in x, and decreasing in y, $\forall\ x, y \in (0,\infty)$ and
(μ, m) is a solution of the system

$$f(m,\mu) = m \ \ \textit{and} \ \ f(\mu,m) = \mu$$

then $\mu = m$.
By using Theorem (3.4.1), the equilibrium point $\bar{y} = 0$ is globally attractive.

□

Theorem 4.3.8. *Every oscillatory solution of Eq.* (4.3.28) *has semi-cycles of length at least k.*

Proof. As the function $f(x,y)$ is increasing in x, and decreasing in y,
so the proof follows immediately from Theorem (3.3.3). □

4.3.11 The Case $\alpha = \gamma = \mathbf{A} = 0$: $x_{n+1} = \frac{\beta x_n}{Bx_n + Cx_{n-k}}$

Lemma 4.3.9. *The change of variables* $x_n = \frac{\beta}{Cy_n}$ *reduces Eq.* (4.3.15) *to the difference equation*

$$y_{n+1} = p + \frac{y_n}{y_{n-k}} \tag{4.3.31}$$

where $p = \frac{B}{C}$ *and the initial conditions* $y_{-k}, \dots, y_0$ *are nonnegative real numbers.*

Proof. Substitute $x_n = \frac{\beta}{Cy_n}$ in Eq. (4.3.15)
we get,

$$\frac{\beta}{Cy_{n+1}} = \frac{\beta \frac{\beta}{Cy_n}}{B\frac{\beta}{Cy_n} + \frac{\beta}{y_{n-k}}}$$

then

$$\frac{1}{y_{n+1}} = \frac{\frac{1}{y_n}}{\frac{B}{Cy_n} + \frac{1}{y_{n-k}}}$$

$$\Longrightarrow y_{n+1} = \frac{B}{C} + \frac{y_n}{y_{n-k}}.$$

By setting $p = \frac{B}{C}$, we get Eq. (4.3.31).

□

The Eq. (4.3.31) was investigated by M.Saleh and M.Aloqeili in [20]

4.3.12 The Case $\alpha = \beta = \mathbf{A} = 0$: $x_{n+1} = \frac{\gamma x_{n-k}}{Bx_n + Cx_{n-k}}$

Lemma 4.3.10. *The change of variables* $x_n = \frac{\gamma}{By_n}$ *reduces Eq.* (4.3.16) *to the difference equation*

$$y_{n+1} = p + \frac{y_{n-k}}{y_n} \tag{4.3.32}$$

where $p = \frac{C}{B}$ *and the initial conditions* $y_{-k}, \dots, y_0$ *are non negative real numbers.*

Proof. Substitute $x_n = \frac{\gamma}{By_n}$ in Eq. (4.3.16)
we get,

$$\frac{\gamma}{By_{n+1}} = \frac{\gamma \frac{\gamma}{By_{n-k}}}{B\frac{\gamma}{By_n} + C\frac{\gamma}{By_{n-k}}}$$

then

$$\frac{1}{y_{n+1}} = \frac{\frac{1}{y_{n-k}}}{\frac{C}{By_{n-k}} + \frac{1}{y_n}}$$

$$\Longrightarrow y_{n+1} = \frac{C}{B} + \frac{y_{n-k}}{y_n}$$

set $p = \frac{C}{B}$, we get Eq. (4.3.32).

□

The Eq. (4.3.32) was studied in [15] by W.He and X.X.Yan, also in [19] by M.Saleh and Aloqeili.

4.3.13 The Case $\alpha = \beta =$C = 0 : $x_{n+1} = \frac{\gamma x_{n-k}}{A+Bx_n}$

Lemma 4.3.11. *The change of variables* $x_n = \frac{\gamma}{B} y_n$ *reduces Eq.* (4.3.17) *to the difference equation*

$$y_{n+1} = \frac{y_{n-k}}{p + y_n} \tag{4.3.33}$$

where $p = \frac{A}{\gamma}$*, and the initial conditions* $y_{-k}, \ldots, y_0$ *are arbitrary nonnegative real numbers.*

Proof. Substitute $x_n = \frac{\gamma}{B} y_n$ in Eq. (4.3.17), to get

$$\frac{\gamma}{B} y_{n+1} = \frac{\frac{\gamma\gamma}{B} y_{n-k}}{A + \frac{B\gamma}{B} y_n}$$

then

$$y_{n+1} = \frac{\gamma y_{n-k}}{\gamma \left[\frac{A}{\gamma} + y_n \right]}$$

set $p = \frac{A}{\gamma}$ to get Eq. (4.3.33). □

The Eq. (4.3.33) has two equilibrium points $\bar{y} = 0$ *and* $\bar{y} = 1 - p$.

And the linearized equation

$$z_{n+1} + \frac{\bar{y}}{(p+\bar{y})^2} z_n - \frac{(p+\bar{y})}{(p+\bar{y})^2} z_{n-k} = 0.$$

Theorem 4.3.9. *The equilibrium point $\bar{y} = 0$ is locally stable when $p > 1$.*

The proof follows immediately from Theorem (3.2.2).

Theorem 4.3.10. *The Eq.* (4.3.33) *has no solution of prime period two.*

Proof. The proof is the same as proof Theorem (4.3.6), so is omitted.

□

Theorem 4.3.11. *Assume that $p > 1$ then the equilibrium point $\bar{y} = 0$ of Eq.* (4.3.33) *is globally attractive.*

Proof. Let the function

$$f(x, y) = \frac{y}{p + x}$$

where $f : (0, \infty) \times (0, \infty) \longrightarrow (0, \infty)$ is continuous function, as $f(x, y)$ is decreasing in x, and increasing in y, $\forall\ x, y \in (0, \infty)$ and the difference equation has no solution of prime period two in $(0, \infty)$. Then by using Theorem (3.4.3), the equilibrium point $\bar{y} = 0$ is global attractive.

□

Theorem 4.3.12. *Every oscillatory solution of Eq.* (4.3.33) *has semi-cycles of length k.*

Proof. As the function $f(x, y)$ is decreasing in x, and increasing in y, so the proof follows immediately from Theorem (3.3.4). □

4.3.14 The Case $\alpha = \beta =$ B = 0 : $x_{n+1} = \frac{\gamma x_{n-k}}{A + C x_{n-k}}$

Lemma 4.3.12. *The change of variables $x_n = \frac{\gamma}{C} y_n$ reduces Eq.* (4.3.18) *to the difference equation*

$$y_{n+1} = \frac{y_{n-k}}{p + y_{n-k}} \qquad (4.3.34)$$

where $p = \frac{A}{\gamma}$, and the initial conditions $y_{-k}, \ldots, y_0$ are arbitrary nonnegative real numbers.

Proof. Substitute $x_n = \frac{\gamma}{C} y_n$ in Eq. (4.3.18), to get

$$\frac{\gamma}{C} y_{n+1} = \frac{\frac{\gamma\gamma}{C} y_{n-k}}{A + \frac{C\gamma}{C} y_{n-k}}$$

then

$$y_{n+1} = \frac{\gamma y_{n-k}}{\gamma \left[\frac{A}{\gamma} + y_{n-k}\right]}$$

set $p = \frac{A}{\gamma}$ to get Eq. (4.3.34) □

The Eq. (4.3.34) has two equilibrium points $\bar{y} = 0$ *and* $\bar{y} = 1 - p$.

And the linearized equation

$$z_{n+1} - \frac{p}{(p + \bar{y})^2} z_{n-k} = 0.$$

Theorem 4.3.13. *Assume that $p > 1$ then the equilibrium point $\bar{y} = 0$ is locally stable.*

The proof follows immediately from Theorem (3.2.2).

Theorem 4.3.14. *The Eq.* (4.3.34) *has no solution of prime period two.*

Proof. Let

$$\ldots, \phi, \psi, \phi, \psi, \ldots$$

be a period two solution of Eq. (4.3.34), where ϕ *and* ψ are two arbitrary positive and distinct real numbers.

- If k is odd, then $y_{n+1} = y_{n-k}$ and $\phi,\ \psi$ satisfy the following systems:

$$\phi = \frac{\phi}{p + \phi} \quad \textit{and} \quad \psi = \frac{\psi}{p + \psi} \tag{4.3.35}$$

  simplifying the relation in Eq. (4.3.35) to get,

$$(\phi - \psi)\left[p + (\phi + \psi) - 1\right] = 0$$

$\Longrightarrow \phi + \psi = 1 - p$ when $p > 1$ then $\phi + \psi$ is negative and this impossible.

So $\phi = \psi$.

- If k is even, then $y_n = y_{n-k}$ and $\phi,\ \psi$ satisfy the following systems:

$$\phi = \frac{\psi}{p + \psi} \quad \textit{and} \quad \psi = \frac{\phi}{p + \phi}. \tag{4.3.36}$$

Simplifying Eq. (4.3.36), we obtain

$$(\phi - \psi)\,[p + 1] = 0$$

as $p \neq -1$, so $\phi = \psi$.

The proof is complete.

□

4.4 Four Parameters are zero

In this section we will study the character of solution of Eq. (3.1.1) where four parameters are zero. There are (9)cases for this equation, namely:

$$x_{n+1} = \frac{\gamma x_{n-k}}{C x_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{4.4.1}$$

$$x_{n+1} = \frac{\alpha}{A}, \qquad n = 0, 1, 2, \ldots \tag{4.4.2}$$

$$x_{n+1} = \frac{\beta x_n}{B x_n}, \qquad n = 0, 1, 2, \ldots \tag{4.4.3}$$

$$x_{n+1} = \frac{\alpha}{B x_n}, \qquad n = 0, 1, 2, \ldots \tag{4.4.4}$$

$$x_{n+1} = \frac{\alpha}{C x_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{4.4.5}$$

$$x_{n+1} = \frac{\beta x_n}{A}, \qquad n = 0, 1, 2, \ldots \tag{4.4.6}$$

$$x_{n+1} = \frac{\beta x_n}{C x_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{4.4.7}$$

$$x_{n+1} = \frac{\gamma x_{n-k}}{A}, \qquad n = 0, 1, 2, \ldots \tag{4.4.8}$$

$$x_{n+1} = \frac{\gamma x_{n-k}}{B x_n}, \qquad n = 0, 1, 2, \ldots \tag{4.4.9}$$

Where the parameters $\alpha,\ \beta,\ \gamma\ and\ A,\ B,\ C$ are non-negative real numbers and the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are arbitrary real numbers, and the denominator is nonzero.

Of these nine equations, Eqs. (4.4.1), (4.4.2) and (4.4.3) are trivial.

Eqs. (4.4.6) and (4.4.8) are linear difference equations.

Every solution of Eq. (4.4.4) is periodic with period two, and every solution of Eq. (4.4.5) is periodic with period $2(k+1)$. So, we just study Eqs. (4.4.7) and (4.4.9).

4.4.1 The case $\alpha = \gamma = \mathbf{A} = \mathbf{B} = 0$: $x_{n+1} = \frac{\beta x_n}{C x_{n-k}}$

By the change of variables $x_n = \frac{\beta}{C}\, e^{y_n}$ Eq. (4.4.7) reduces to the difference equation

$$y_{n+1} = y_n - y_{n-k} \tag{4.4.10}$$

when $k = 1$, then every positive solution of Eq. (4.4.10) is periodic with period six, and its solution is:

$$\ldots, x_{-1}, x_0, \frac{x_0}{x_{-1}}, \frac{1}{x_{-1}}, \frac{1}{x_0}, \frac{x_{-1}}{x_0}, \ldots$$

also, when $k = 1$, the following difference equation

$$y_{n+1} + y_{n-k} - y_n = 0$$

has a general solution

$$y_n = (1)^n \left[c_1 \cos\frac{n\pi}{3} + c_2 \sin\frac{n\pi}{3}\right]$$

where $r = 1$, and $\theta = \frac{\pi}{3}$.

Lemma 4.4.1. *The equilibrium point of Eq.* (4.4.10) *is unstable when* $k \geq 2$.

The proof is consequently from Theorem (3.2.2).

4.4.2 The case $\alpha = \beta = \mathbf{C} = \mathbf{A} = 0$: $x_{n+1} = \frac{\gamma x_{n-k}}{B x_n}$

The change of variables $x_n = \frac{\gamma}{B}\, e^{y_n}$ reduces Eq. (4.4.9) to the difference equation

$$y_{n+1} - y_{n-k} + y_n \quad n = 0, 1, \ldots \tag{4.4.11}$$

this transformation, substitute $x_n = \frac{\gamma}{B}\, e^{y_n}$ in Eq. (4.4.9), so we get

$$\frac{\gamma}{B}\, e^{y_{n+1}} = \frac{\gamma}{B}\frac{e^{y_{n-k}}}{e^{y_n}}$$

cancel $\frac{\gamma}{B}$ from both side, we obtain

$$e^{y_{n+1}} = \frac{e^{y_{n-k}}}{e^{y_n}}$$

then

$$e^{y_{n+1}} = e^{y_{n-k}} e^{-y_n} \Longrightarrow y_{n+1} - y_{n-k} + y_n$$

when $k = 1$, then the following difference equation

$$y_{n+1} - y_{n-k} + y_n = 0$$

has a general solution

$$y_n = c_1 \left(\frac{-1+\sqrt{5}}{2} \right)^n + c_2 \left(\frac{-1-\sqrt{5}}{2} \right)^n$$

Lemma 4.4.2. *The equilibrium point of Eq.* (4.4.11) *is unstable when* $k \geq 2$.

The proof is consequently from Theorem (3.2.2).

4.5 Numerical Analysis

To illustrate the results of the previous chapters and to support our theoretical discussion, we will consider a few numerical examples in this section. These examples represent different types of qualitative behavior of solutions to nonlinear difference equations.

Example 4.5.1. *Consider the third order difference equation when k=2 in Eq.* (3.2.1)*:*

$$y_{n+1} = \frac{p + y_n + Ly_{n-k}}{q + y_n + dy_{n-k}}.$$

And assume that p=9, q=5, L=3 and d=4. So the equation will be reduced to the following:

$$y_{n+1} = \frac{9 + y_n + 3y_{n-2}}{5 + y_n + 4y_{n-2}}.$$

We assume the initial points $\{y_{-2}, y_{-1}, y_0\}$ are $\{.3, .1, .8\}$.
Then, the results is below.

```
>> diffequation
           ---------

 First: Input The Constants Values Of Your Difference Equation

The value of the positive parameters p= 9\\
The value of the positive parameters l= 3\\
The value of the positive parameters q= 5\\
 The value of the positive parameters d=4
           ---------
Second: Input The value of k
 k= 2
           ---------

 Third: Enter the initial conditions of Diff.Equation
```

```
Enter the value of y
 y=.3
Enter the value of y
 y=.1
Enter the value of y
 y=.8
            -------
The results are:
```

n	y(n)	n	y(n)	n	y(n)	n	y(n)
1.0000	0.3000	26.0000	1.6016	51.0000	1.6026	76.0000	1.6026
2.0000	0.1000	27.0000	1.6000	52.0000	1.6027	77.0000	1.6026
3.0000	0.8000	28.0000	1.6124	53.0000	1.6026	78.0000	1.6026
4.0000	4.6522	29.0000	1.6026	54.0000	1.6026	79.0000	1.6026
5.0000	2.7080	30.0000	1.6040	55.0000	1.6026	80.0000	1.6026
6.0000	2.1032	31.0000	1.5975	56.0000	1.6026	81.0000	1.6026
7.0000	0.9880	32.0000	1.6030	57.0000	1.6026	82.0000	1.6026
8.0000	1.2467	33.0000	1.6019	58.0000	1.6026	83.0000	1.6026
9.0000	1.4075	34.0000	1.6054	59.0000	1.6026	84.0000	1.6026
10.0000	2.1066	35.0000	1.6023	60.0000	1.6026	85.0000	1.6026
11.0000	1.7802	36.0000	1.6030	61.0000	1.6026	86.0000	1.6026
12.0000	1.7014	37.0000	1.6012	62.0000	1.6026	87.0000	1.6026
13.0000	1.3913	38.0000	1.6029	63.0000	1.6026	88.0000	1.6026
14.0000	1.5285	39.0000	1.6024	64.0000	1.6026	89.0000	1.6026
15.0000	1.5577	40.0000	1.6034	65.0000	1.6026	90.0000	1.6026
16.0000	1.7303	41.0000	1.6024	66.0000	1.6026	91.0000	1.6026
17.0000	1.6341	42.0000	1.6028	67.0000	1.6026	92.0000	1.6026
18.0000	1.6245	43.0000	1.6022	68.0000	1.6026	93.0000	1.6026
19.0000	1.5391	44.0000	1.6028	69.0000	1.6026	94.0000	1.6026
20.0000	1.5903	45.0000	1.6026	70.0000	1.6026	95.0000	1.6026
21.0000	1.5921	46.0000	1.6028	71.0000	1.6026	96.0000	1.6026
22.0000	1.6376	47.0000	1.6025	72.0000	1.6026	97.0000	1.6026
23.0000	1.6069	48.0000	1.6027	73.0000	1.6026	98.0000	1.6026
24.0000	1.6079	49.0000	1.6025	74.0000	1.6026	99.0000	1.6026
25.0000	1.5844	50.0000	1.6027	75.0000	1.6026	100.0000	1.6026

```
            -------
Some Analysis Of The Results:

 1- As K is even, there is no positive period two solutions

2- Since (p+l)>(q+d), p >q & d >l Then The Following Are True:

 (a) The Equilibrium point  is asymptotically stable
 (b)   =    =    = is globally asymptotically stable
        -------
                  The End
```

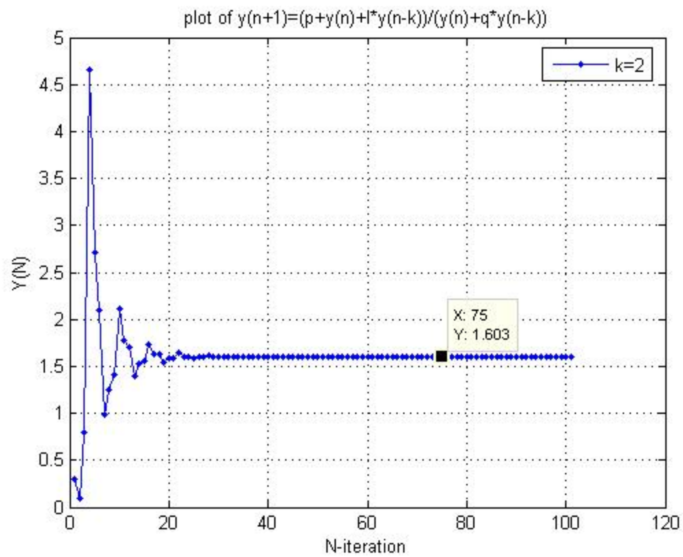

Figure 4.5.1: Plot of $y_{n+1} = \frac{9+y_n+3y_{n-2}}{5+y_n+4y_{n-2}}$

Example 4.5.2. *Consider the second order difference equation when k=1 in Eq.* (3.2.1)*:*

$$y_{n+1} = \frac{p + y_n + Ly_{n-k}}{q + y_n + dy_{n-k}}.$$

And assume that p=2, q=7, L=6 and d=1. So the equation will be reduced to the following:

$$y_{n+1} = \frac{2 + y_n + 6y_{n-1}}{7 + y_n + 1y_{n-1}}.$$

We assume the initial points $\{y_{-1}, y_0\}$ are $\{.9, 2.3\}$.
Then, the results is below.

```
>> diffequation
            ---------

 First: Input The Constants Values Of Your Difference Equation
```

```
The value of the positive parameters p= 2 \\
The value of the positive parameters l= 6 \\
The value of the positive parameters q= 7\\
 The value of the positive parameters d= 1
           ---------
Second: Input The value of k
 k= 1
           ---------

 Third: Enter the initial conditions of Diff.Equation

Enter the value of y
 y=.9
Enter the value of y
 y=2.3
            -------
The results are:
```

n	y(n)	n	y(n)	n	y(n)	n	y(n)
1.0000	0.9000	26.0000	1.1019	51.0000	1.1019	76.0000	1.1019
2.0000	2.3000	27.0000	1.1019	52.0000	1.1019	77.0000	1.1019
3.0000	1.1279	28.0000	1.1019	53.0000	1.1019	78.0000	1.1019
4.0000	0.9826	29.0000	1.1019	54.0000	1.1019	79.0000	1.1019
5.0000	1.0982	30.0000	1.1019	55.0000	1.1019	80.0000	1.1019
6.0000	1.1276	31.0000	1.1019	56.0000	1.1019	81.0000	1.1019
7.0000	1.1023	32.0000	1.1019	57.0000	1.1019	82.0000	1.1019
8.0000	1.0970	33.0000	1.1019	58.0000	1.1019	83.0000	1.1019
9.0000	1.1019	34.0000	1.1019	59.0000	1.1019	84.0000	1.1019
10.0000	1.1028	35.0000	1.1019	60.0000	1.1019	85.0000	1.1019
11.0000	1.1019	36.0000	1.1019	61.0000	1.1019	86.0000	1.1019
12.0000	1.1017	37.0000	1.1019	62.0000	1.1019	87.0000	1.1019
13.0000	1.1019	38.0000	1.1019	63.0000	1.1019	88.0000	1.1019
14.0000	1.1019	39.0000	1.1019	64.0000	1.1019	89.0000	1.1019
15.0000	1.1019	40.0000	1.1019	65.0000	1.1019	90.0000	1.1019
16.0000	1.1019	41.0000	1.1019	66.0000	1.1019	91.0000	1.1019
17.0000	1.1019	42.0000	1.1019	67.0000	1.1019	92.0000	1.1019
18.0000	1.1019	43.0000	1.1019	68.0000	1.1019	93.0000	1.1019

```
   19.0000    1.1019   44.0000    1.1019   69.0000    1.1019   94.0000    1.1019
   20.0000    1.1019   45.0000    1.1019   70.0000    1.1019   95.0000    1.1019
   21.0000    1.1019   46.0000    1.1019   71.0000    1.1019   96.0000    1.1019
   22.0000    1.1019   47.0000    1.1019   72.0000    1.1019   97.0000    1.1019
   23.0000    1.1019   48.0000    1.1019   73.0000    1.1019   98.0000    1.1019
   24.0000    1.1019   49.0000    1.1019   74.0000    1.1019   99.0000    1.1019
   25.0000    1.1019   50.0000    1.1019   75.0000    1.1019  100.0000    1.1019

            -------
Some Analysis Of The Results:

 1- The Equilibrium Point  Of This Equation =1.0000
there is a prime period two solution
            -------
                  The End
>>
```

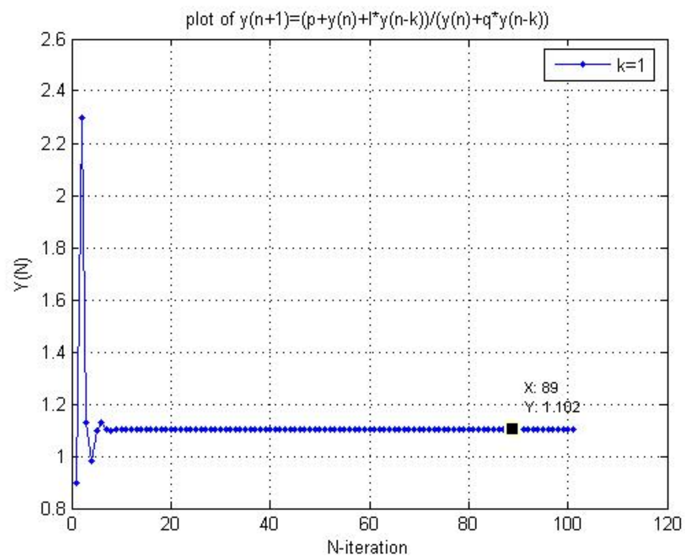

Figure 4.5.2: Plot of $y_{n+1} = \frac{2+y_n+6y_{n-1}}{7+y_n+1y_{n-1}}$

Example 4.5.3. *Consider the fourth order difference equation when k=3 in Eq.* (3.2.1)*:*

$$y_{n+1} = \frac{p + y_n + Ly_{n-k}}{q + y_n + dy_{n-k}}.$$

And assume that p=6, q=5, L=7 and d=10. So the equation will be reduced to the following:

$$y_{n+1} = \frac{6 + y_n + 7y_{n-3}}{5 + y_n + 10y_{n-3}}.$$

We assume the initial points $\{y_{-3}, y_{-2}, y_{-1}, y_0\}$ are $\{1, .4, .28, 0\}$.
Then, the results is below.

```
>> diffequation
            ---------

 First: Input The Constants Values Of Your Difference Equation
```

```
The value of the positive parameters p= 6
 The value of the positive parameters l= 7
 The value of the positive parameters q= 5
The value of the positive parameters d= 10
           ---------
Second: Input The value of k
 k= 3
           ---------

 Third: Enter the initial conditions of Diff.Equation

Enter the value of y
 y=1
Enter the value of y
 y=.4
Enter the value of y
 y=.28
Enter the value of y
 y=0
            -------
The results are:
```

n	y(n)	n	y(n)	n	y(n)	n	y(n)
1.0000	1.0000	26.0000	1.8686	51.0000	1.8685	76.0000	1.8685
2.0000	0.4000	27.0000	1.8683	52.0000	1.8685	77.0000	1.8685
3.0000	0.2800	28.0000	1.8682	53.0000	1.8685	78.0000	1.8685
4.0000	0	29.0000	1.8687	54.0000	1.8685	79.0000	1.8685
5.0000	2.6000	30.0000	1.8685	55.0000	1.8685	80.0000	1.8685
6.0000	2.4783	31.0000	1.8686	56.0000	1.8685	81.0000	1.8685
7.0000	2.6915	32.0000	1.8686	57.0000	1.8685	82.0000	1.8685
8.0000	3.2293	33.0000	1.8685	58.0000	1.8685	83.0000	1.8685
9.0000	1.6901	34.0000	1.8685	59.0000	1.8685	84.0000	1.8685
10.0000	1.7781	35.0000	1.8685	60.0000	1.8685	85.0000	1.8685
11.0000	1.7471	36.0000	1.8685	61.0000	1.8685	86.0000	1.8685
12.0000	1.6963	37.0000	1.8685	62.0000	1.8685	87.0000	1.8685
13.0000	1.9244	38.0000	1.8685	63.0000	1.8685	88.0000	1.8685
14.0000	1.8836	39.0000	1.8685	64.0000	1.8685	89.0000	1.8685

15.0000	1.8941	40.0000	1.8685	65.0000	1.8685	90.0000	1.8685
16.0000	1.9053	41.0000	1.8685	66.0000	1.8685	91.0000	1.8685
17.0000	1.8544	42.0000	1.8685	67.0000	1.8685	92.0000	1.8685
18.0000	1.8665	43.0000	1.8685	68.0000	1.8685	93.0000	1.8685
19.0000	1.8634	44.0000	1.8685	69.0000	1.8685	94.0000	1.8685
20.0000	1.8613	45.0000	1.8685	70.0000	1.8685	95.0000	1.8685
21.0000	1.8721	46.0000	1.8685	71.0000	1.8685	96.0000	1.8685
22.0000	1.8687	47.0000	1.8685	72.0000	1.8685	97.0000	1.8685
23.0000	1.8696	48.0000	1.8685	73.0000	1.8685	98.0000	1.8685
24.0000	1.8699	49.0000	1.8685	74.0000	1.8685	99.0000	1.8685
25.0000	1.8677	50.0000	1.8685	75.0000	1.8685	100.0000	1.8685

```
            -------
Some Analysis Of The Results:

- Since (p+l)<(q+d), p >q & d >l Then The Following Are True:

 (a) The Equilibrium point  is asymptotically stable

 (b)   =    =    = is globally asymptotically stable

             -------
                  The End
>>
```

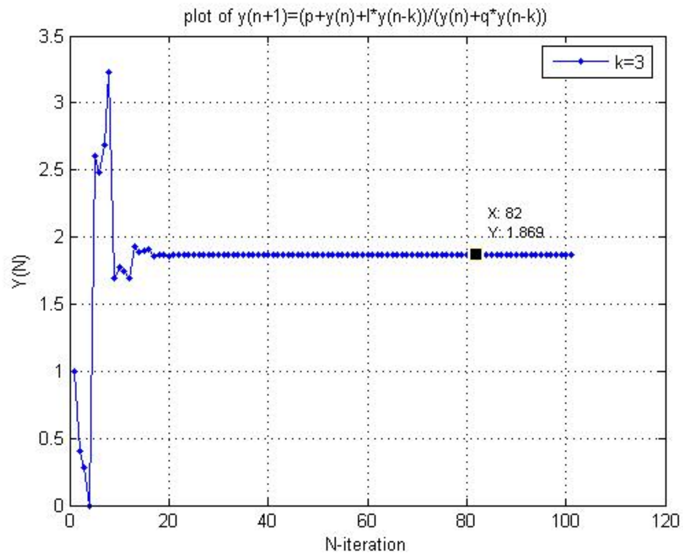

Figure 4.5.3: Plot of $y_{n+1} = \frac{6+y_n+7y_{n-3}}{5+y_n+10y_{n-3}}$

5 Matlab Code 7.1

The mfile function investigate the nonlinear rational difference equation:

$$y_{n+1} = \frac{p + y_n + Ly_{n-k}}{q + y_n + dy_{n-2}}.$$

Where the parameters p, q, L and d and the initial conditions are non negative real numbers.

We create this file to find the computational solution and to compare between the theoretical approach, and computational approach.

```
%Dynamical Of Non Linear Difference Equation
%Ayah Asad
%program 1
clear all
 format short
%#######################

disp('           ---------                   ');
```

```
 fprintf('\n
First: Input The Constants Values Of Your Difference Equation \n')
 disp('                        ')
 p=input('The value of the positive parameters p= ');
 l=input('The value of the positive parameters l= ');
 q=input('The value of the positive parameters q= ');
  d=input('The value of the positive parameters d= ');

disp('           ---------                   ');

k=input('Second: Input The value of k \n k= ');
 disp('          ---------                      ');
fprintf(1,'\n Third: Enter the initial conditions of the
Diff.Equation\n ')
 disp('    ');
%########################
ans=pqdlk(p,q,d,l,k); disp('            -------         ')
disp('The results are: ')
disp('_____________________________________________________________________________
disp('     n       y(n)      n         y(n)       n      y(n)   n
y(n)')
disp(______________________________________________________________________________
D=[ans(1:25,:),ans(26:50,:),ans(51:75,:),ans(76:100,:)]; disp(D)
disp('           -------        ');
 disp('Some Analysis Of The Results:')
if rem(k,2)==0
 fprintf('\n 2- As K is even ,there is no positive period two solutions \n ')
else

    if l<(1+q)
        if d >1
fprintf('\n 2- As l<1+q ,d>1 & K is odd, there is no positive
period two solution \n')

        else
    fprintf('there is a prime period two solution')
        end
    end
```

```
end
 EI=(p+l)/(d+q);
 disp('                                ')
if (p+l)>(q+d)
    if p>q
        if d > l
    disp('3- Since (p+l)>(q+d), p >q & d >l')

    disp('Then The Following is True:')
    fprintf('\n (b)   =    =    = is globally asymptotically stable ')
fprintf('\n (b)   =    =    = is locally asymptotically stable ')
    end
    end
end
%#########################
if(p+l)<(q+d)
    if p>q
        if d > l
    disp('3- Since (p+l)<(q+d), p >q & d >l')

    disp('Then The Following are True:')
    fprintf('\n (b)   =    =    = is globally asymptotically stable \n')
     fprintf('\n (b)   =    =    = is locally asymptotically stable \n')
    end
    end
end

   disp('            -------         ')
disp('                  The End                   ')


% ####################
```

```
 function ans=pqdlk(p,q,d,l,k); for i=1:k+1;
    y(i)=input ('Enter the value of y \n y=');
end

for  n=k+1:100;
 y(n+1)=(p+y(n)+l*y(n-k))/(y(n)+q*y(n-k));
 y(n+1);
end t=1:101;
 ans=[t;y]';
plot(t,y,'b.-') xlabel('N-iteration'); ylabel('Y(N)');
 title('plot of y(n+1)=(p+y(n)+l*y(n-k))/(y(n)+q*y(n-k)) ')
 hold on
 grid on
p1=strcat('k=',num2str(k));
 legend(p1)
```

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